FREE VIBRATION ANALYSIS OF LAMINATED COMPOSITE PLATES

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CERTIFICATE

It is certified that the work contained in the thesis entitled "FREE VIBRATION ANALYSIS OF LAMINATED COMPOSITE PLATES", by "SAMPATH, N. P.", has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

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IIT Kanpur

To appa, amma, my family, my teachers and to Pope and Shankar for all their love and faith in me.

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NOMENCLATURE

a - Dimension of plate in X direction.

b - Dimension of plate in Y direction.

b_i, c_i - Constants of the i^{th} term of the Ritz series.

C,C2,C3 - Notation for clamped boundary condition.

$C^u_{ij}, C^v_{ij}, C^w_{ij}$ - Coefficients of in-plane and out-of-plane displacements along X, Y, and Z.

E_T - Minor Young's modulus of a lamina.

$\{f\}$ - Load vector.

$\{g\}$ - Orthonormal set of vectors.

h - Plate thickness.

I_o - Lagrangian time integral.

$[I]$ - Identity matrix.

$[K]$ - Stiffness matrix.

$[l]$ - Linearly independent set of vectors.

L - Longitudnal direction of fiber.

L - Lagrangian function.

$[M]$ - Mass matrix.

n - Direction normal to plate edge.

R - Residual or error.

S,S2,S3 - Notation for simply-supported boundary condition.

t - Direction tangential to plate edge.

t_1, t_2 - Upper and lower time limits.

T - Transverse direction of fiber.

T - Kinetic energy of the system.

u, v, w - Displacements along X, Y, and Z directions.

u_o, v_o, w_o - Mid-plane displacements of the plate.

V - Potential energy of the system.

Z_n - Distance of n^{th} lamina from plate mid-plane.

$\{\alpha\}, \{\beta\}, \{\phi\}$ - Terms of the Ritz series.

δ - Variator.

$\{\varepsilon\}$ - Linearly independent basis set.

ξ, η - Normalised Coordinates.

ω - Natural frequency.

ABSTRACT

Laminated composite plates have been analysed for their vibrational characteristics using a new technique based on the Gram-Schmidt procedure, to generate the orthonormal bases for the series expansion of the displacements, in the Rayleigh-Ritz method. The technique has been demonstrated to be very powerful for the analysis of moderately thick plates even shear deformation is neglected. The rapid convergence, the ease with which starting terms maybe chosen and the simple Gaussian integration possible, make the present technique extremely useful. Finally, the technique has been effectively used to analyse free vibrations of tapered plates.

CHAPTER 1

INTRODUCTION AND LITERATURE SURVEY

[1.1] Introduction:

The primary load carrying and supporting framework of any system, be it a building, an automobile or an aircraft, is its structure. In particular, Aerospace structures have always afforded the greatest of challenges to the designer due to the innumerable constraints he has to work with. Any Aerospace structural design has to satisfy - apart from its main functional requirement - several other criteria such as maximum strength to weight and stiffness to weight ratios', high fatigue strength, and good reliability.

In order to be able to predict the total response of a structure to any input, a knowledge of the responses of individual structural elements that go to make up the overall structure, is essential. Some typical structural elements are solid and thin-walled beams, plates, shells, struts and the like. The response of these elements with varying degrees of complexity of material, loads and boundary conditions needs to be fully understood before an actual design can be attempted.

The advent of Composite materials, has revolutionized structural design, bringing with it many advantages. Composites consist essentially of high strength -continuous or short- fibers dispersed in a matrix of relatively lower strength. They

can be tailored to suit a given application. This can be done by choosing suitable combinations of fiber/matrix materials and orienting the high strength, low density fibers along preferred directions. Features like reduced weight, higher strength and stiffness, directional properties and so on can be easily achieved through composites.

However, composites are generally anisotropic in nature and hence introduce complicating effects like in-plane-shear coupling, bending-stretching coupling and so on. Nevertheless, the effects are partially offset by exploiting symmetry in the geometry of the structure.

Laminated composite plate elements form an important part of any aerospace structure. A thorough dynamic analysis would reveal important vibrational and damping characteristics of such elements and this could form a vital input to the designer. This is necessary since the stresses induced due to dynamic loads can be an order of magnitude higher than static loads.

A detailed review of previous research work on vibration analysis of plates reveals that most of the researchers' have used trigonometric or hyperbolic admissible functions (involving tedious integrations for the stiffness and mass matrix elements) in the Rayleigh-Ritz method or have used the Finite Element method. Very few have attempted to use simple algebraic orthonormal polynomials. A much lesser number have applied the Ritz technique with simple algebraic admissible functions to the vibration analysis of Laminated composite

plates.

In the present work, rectangular Laminated composite plates have been analysed for free vibrational characteristics. The assumed displacement field corresponds to that of the Classical Plate Theory. The Rayleigh-Ritz technique has been employed and the orthonormal bases for the Ritz series expansion of the displacements have been generated by the Gram-Schmidt procedure.

[1.2] Literature Survey:

In accordance with the conclusions drawn from the survey of available literature on plate vibrations, as has just been mentioned, it is only appropriate that the vast spectrum of research work in this field be divided into two distinct categories. They are as follows:

▸ i) Dynamic analysis wherein conventional admissible functions such as trigonometric and hyperbolic beam characteristic functions have been used in a Ritz technique or the Finite Element method has been employed.

The work of Reissner and Stavsky[1] provided the theoretical base for the study of vibration and buckling of laminated composite plates. They were one of the first few to consider the bending-stretching coupling effects in composites, in their analysis.

Bert and Mayberry[2] used trigonometric and hyperbolic

functions for free vibration analysis of un-symmetrically laminated composite plates with all edges clamped. They have also presented corresponding experimental values for the first few frequencies.

An early and notable effort in anisotropic plate analysis, galvinised by the coming of Composite materials, was that due to Ashton and Waddoups[3]. They demonstrated the use of beam eigenfunctions together with the Ritz method in vibration analyses of rectangular anisotropic plates.

Whitney and Leissa[4] presented closed form, exact solutions of free vibration frequencies and mode shapes for anti-symmetric cross-ply and angle-ply laminates with simply-supported boundary conditions. Considering the classical types of boundary conditions (Clamped, Simply-supported or Free) that could exist on each edge of an un-symmetrically laminated plate, the number of independent problems possible, are many. In fact 16 different possibilities can be identified for each edge. One such possibility of a plate with simple supports ($N_{nt} = 0$ instead of $u_t = 0$ in the simply-supported boundary condition) was analysed by Whitney and Leissa[5]. They used a generalised Fourier series.

Later Whitney[6] showed, using clamped boundary conditions for square anisotropic plates, how a generalised Fourier series solution led to an infinite frequency determinant different from that of the Ritz approach yielding more rapidly converging results.

In an excellent paper on buckling and vibration, Jones[7]

presented an exact solution and results for simply-supported plates, un-symmetrically laminated about their middle surface. Jones showed that bending-extension coupling due to asymmetry reduced vibration frequencies for common Composites like Boron/Epoxy and Graphite/Epoxy. It was also shown that for anti-symmetric laminates, the effect of coupling died out rapidly with increasing number of layers but that for generally unsymmetric laminates the decay was slower.Simple trigonometric functions were used for the displacements. The largest effects were shown to occur for high modulus ratio materials.

Lin and King[8] showed that exact solutions could be obtained for a class of unsymmetrically laminated plates having two opposite sides simply-supported with arbitrary edge conditions on the other two sides, by suitably choosing the admissible functions.

Bert[9] derived a relation to compute frequencies of a generally anisotropic plate having arbitrary boundary conditions, given the frequencies of a corresponding rectangular isotropic plate. Later, the same investigator used this result and suggested an optimal design procedure to maximise the fundamental frequencies of such plates.

The work by Bert and Chen,[10] aimed at studying the effect of shear deformation on vibration of anti-symmetric angle-ply laminates. They obtained closed form solutions for all edges simply-supported, using the YNS (Yang-Norris-Stavsky) theory. They too, employed trigonometric functions for the displacement and slope variables. A high modulus (Graphite/Epoxy) material

with $E_L/E_T = 40$ was used and results were presented showing the effects of plate aspect ratio, length/thickness ratio, lamination angle and number of layers. It was also shown that neglecting in-plane and rotatory inertias had little effect on the fundamental frequency, even for thick plates.

Dickinson[11] initiated the use of the Simply-supported plate functions which were described as admissible functions, derived from the mode shapes of vibration, of plates having two parallel edges simply-supported, the boundary conditions on the other two edges being chosen appropriately for the plate under consideration. It was shown that accuracies resulting from the use of these functions were superior to those from the use of beam functions and the more complicated Bolotin functions.

Reddy,[12] used the YNS theory in his paper on vibration analysis of anti-symmetric laminated plates which was a close parallel to the one published by Bert and Chen[10].However, here the Finite Element method was employed. The plate was discretized using eight noded Serendipity elements. Results for 30^o and 45^o angle ply laminates for two different modulus ratios' (typical of Graphite/Epoxy) with $E_L/E_T = 40$ and 25 were presented and compared with those obtained by Bert and Chen[10]. Also frequencies obtained for a simply-supported thick Isotropic plate were compared with those due to 3-D linear elasticity theory and that due to Mindlin's.

In a later paper, Dickinson and Li[13] used the previously

described simply-supported plate functions to analyse flexural vibrations of rectangular plates by the Rayleigh-Ritz method and confirmed that improvement in accuracy of the natural frequencies was indeed possible, compared to beam functions. However, it was also concluded that 'Simply-supported' functions were less satisfactory for plates with free edges.

Bhimaraddi and Stevens[14] proposed a two dimensional theory for the analysis of thick homogeneous laminated plates. The cubic order polynomial form for in-plane displacement quantities was maintained ensuring at the same time the more realistic parabolic variation for transverse shear strains. Further, trigonometric displacement functions were used in the analysis.

In another lucid and excellent paper, Lin et al[15] presented their research work with the objective of predicting the natural frequency and Specific Damping Capacity (SDC) of laminated Composite plates in various modes of vibration by the Finite Element method. An 8-noded, 40 degree of freedom element was used. The damped element model was developed by introducing ψ_1, ψ_2 -the SDC of 0° and 90° beam specimen tested in flexure- and ψ_{12}, ψ_{23} -the SDC of 0° and 90° specimen tested in longitudnal shear- in the strain energy expressions.

Experimental results were also presented and the measurement technique was based on the transient testing technique using a zoom-FFT. A micro-computer interfaced with the FFT analyser recorded and calculated the frequency spectrum using resonant curve fitting or half power point technique to

obtain the natural frequency and vibration damping immediately.

Malhotra et al[16] studied the effects of fiber orientation and boundary conditions on the frequencies of thin square orthotropic plates. Comprehensive results for various boundary conditions and lamination angles were presented for four types of Composites: Glass, Kevlar, Boron, Graphite/Epoxy.

▶ ii) Recent work on study of free vibrations of plates using simple algebraic orthonormal polynomials:

Bhat[17] made use of algebraic functions in plate vibration studies. Bhat obtained the natural frequencies of rectangular isotropic plates using a set of beam characteristic orthogonal algebraic polynomials in the Rayleigh-Ritz method. The orthogonal polynomials were generated using the Gram-Schmidt procedure. The first term was so chosen as to satisfy all the boundary conditions of the corresponding beam problems which were a subset of the plate boundary conditions and the succeeding terms satisfied the geometric boundary condition only. Thus the basis for the series expansion of the displacements were formed of products of the x and y polynomial terms so generated, with the procedure automatically ensuring completeness and continuity of these polynomials.

Excellent results for plates with simply-supported, clamped and free edge conditions were presented which compared very well with the exact value and with the results obtained by other investigators.

Dickinson and Blasio[18] made use of the orthogonal

polynomial functions proposed by Bhat[17] for vibration and buckling analysis of isotropic and orthotropic plates, employing the Ritz technique.

The authors confirmed that these simpler polynomials provided an attractive alternative for admissible functions as compared to the conventional trigonometric and hyperbolic functions, such as those used in the 'Simply-supported plate functions' or the 'Degenerated beam functions'.

It was proposed that in the case of plates with free edges, simpler starting terms (of lower orders) could be used with very little loss in accuracy. Vibration and buckling problems of isotropic and specially orthotropic plates, subjected to in-plane direct and shear loading were addressed. Finally, it was concluded that the simple polynomials relaxed the over restraint in the case of plates with free edges and rendered the product integrals for the stiffness and mass matrices, easy.

Bhat[19] proposed a further modification to the approach using algebraic polynomials. Orthogonal polynomials in two variables were used with a little difference in the earlier approach (Bhat[17]) and the new one. These polynomials were used to obtain the natural frequencies of triangular plates of different configurations.

Lam et al[20] presented results of vibration analysis of single layered composite plates. They used a modified version of Bhat's two dimensional orthogonal plate functions, in a Ritz analysis. The effects of boundary conditions, aspect ratio and

fiber orientation on the natural frequencies of Composite plates were studied.

Liew et al[21] used orthogonal plate functions for vibration analysis of rectangular isotropic plates including those with free edges. A starting term similar to that of Bhat[19] was used but with a modified Gram-Schmidt generation of succeeding terms which led to faster convergence. Very good results were obtained for plates with various types of boundary conditions.

Liew and Lam[22] used the orthogonal plate functions (two dimensional) in a Rayleigh-Ritz approach to investigate free vibration of isotropic and anisotropic trapezoidal plates, with simply-supported, clamped and many other types of boundary conditions.

[1.3] Objective and Scope of the Present Work:

The literature review clearly indicates that the technique of using orthogonal algebraic polynomials is an elegant and a very powerful one for the vibration analysis of plates. Further, it is also evident that this technique has, to date, been applied only for the analysis of isotropic and orthotropic plates. Keeping this in view, an application of this technique to the dynamic analysis of general Composite laminates has been attempted in the present work.

The present work focuses on the free vibration analysis of Composite laminates with different boundary conditions,

materials, fiber orientation and lay-ups. The Classical Plate Theory has been used and orthogonal polynomials have been employed in the Rayleigh-Ritz analysis. A computer program (in Fortran 77) that generates the orthonormal polynomials automatically and computes the necessary product integrals has been developed. Simple Gauss Quadrature and numerical differentiation schemes have been used. Eigenvalues and eigenvectors have been computed using available NAG routines - all these on the HP 9000/850s.

Results for two types of materials have been generated for antisymmetric angle-ply and cross-ply and symmetric cross-ply laminates for various boundary conditions. The usual check for the isotropic case has also been made. The results have compared with those generated by standard techniques such as Finite Element techniques and so on. Some results for clamped isotropic tapered plates have also been included.

Chapter 2 covers a detailed description of the problem formulation.

Chapter 3 contains an analysis of the results presented.

Chapter 4 contains conclusions drawn and suggestions for future work.

CHAPTER 2

THEORY AND FORMULATION

[2.1] General Solid Continuum analysis:

[2.1.1] Degrees of freedom of a system:

The minimum number of independent coordinates required to completely determine the configuration of a system.

[2.1.2] Material Point:

A point in the domain of the continuum where a mass point is assumed to be located.

[2.1.3] The Solid Continuum:

A solid continuum is assumed to be a continuous material that is devoid of voids and gaps. The configuration of a deformable solid continuum is determined by the simultaneous positions of its infinite material points relative to a fixed coordinate system. Thus a deformable solid requires specification of an infinite set of degrees of freedom, each corresponding to the position of a material point.

Ideally, the position of a material point is uniquely determined by three independent coordinates. However, it is not practicable to deal with discrete points and hence infinitesimal volumes are considered. These are characterised by one or more field variables such as displacements, slopes and rotations. In turn, each of the field variable is expressed as the sum of an infinite series of linearly independent functions. For example, if Δ is a field variable, we have

$$\Delta = \sum_{i=1} c_i \, \varepsilon_i(x,y,z) \qquad (2.1)$$

The linearly independent basis set $\{\varepsilon_i\}$ spans a configuration space of the system and any subset of it spans a subspace.

In a Dynamic analysis, such as in the free vibration problem, the contemporaneous positions of the material points relative to a fixed coordinate system describe its configuration.

[2.1.4] The Problem:

The objective of any analysis using various techniques such as those described below is to solve for the unknown coefficients c_i, from equations of the type

$$[K]\{c\} = \{f\}$$

for a static problem, or

$$[K]\{c\} = \lambda [M]\{c\}$$

for a dynamic eigenvalue problem, where

$[K]$ is the stiffness matrix

$[M]$ is the mass matrix

$[f]$ is the load vector, and

λ the eigenvalue

The coefficients c_i can be visualised as components of the solution vector in the configuration space for the given set of loads on the system. Thus they determine the extent to which each of the degrees of freedom, ε_i, participate in the solution.

[2.1.5] Some Analytical techniques:

It is obviously impossible to solve for an infinite set of coefficients and hence many of the techniques represent the field variables as the sum of a finite number of terms (say N) of a series as,

$$\Delta = \sum_{i=1}^{N} c_i \, \varepsilon_i \tag{2.2}$$

a) The Rayleigh-Ritz technique:

In the Ritz analysis, each field variable of the problem is expressed, as the sum of a finite number of terms of a series.

The total potential energy, V and the kinetic energy, T, of the system and hence the Lagrangian function (L = T - V), computed by using equations of the type (2.2) in the respective energy expressions, are rendered functions solely of the unknown coefficients, c_i. Thus we have,

$$V = V(c_1, c_2, \ldots, c_M)$$

$$T = T(c_1, c_2, \ldots, c_M) \qquad (2.3)$$

$$L = L(c_1, c_2, \ldots, c_M)$$

Invoking the Principle of Stationary Value of the total Potential and taking a variation on V, we have for a system in equilibrium,

$$\delta V = \left[\frac{\partial V}{\partial c_1} \delta c_1 + \frac{\partial V}{\partial c_2} \delta c_2 + \ldots + \frac{\partial V}{\partial c_M} \delta c_M \right] = 0 \ldots (2.4)$$

Now since the δc_i are all arbitrary and independent, they are all not necessarily equal to zero and so we have,

$$\frac{\partial V}{\partial c_i} = 0 \ , \quad i = 1,2,\ldots,M \quad \ldots\ldots (2.5)$$

Similarly for a system in Dynamic equilibrium undergoing Harmonic oscillations, we can write

$$\left[\frac{\partial \int_{t_1}^{t_2} L \, dt}{\partial c_i} \right] = 0 \quad \ldots\ldots\ldots\ldots (2.6)$$

To ensure that the Ritz approximations result in a solution and that it converges to the true solution as the number of terms of the series are increased, the choice of the set $\{\varepsilon_i\}$ has to satisfy certain conditions are detailed below:

- i) Completeness: The choice of the ε functions should begin from the lowest order of the polynomial being used, preferably from the zeroth order and can increase upto any desired order without intermediate orders being missed.

The effect of ignoring certain intermediate orders, provided the first few terms are complete, is to bias the solution towards certain preferred modes.

Completeness can easily be ensured by choosing complete sets of polynomials from the Pascal's triangle shown below,

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & x & & y & & \\
 & x^2 & & xy & & y^2 & \\
x^3 & & x^2y & & xy^2 & & y^3
\end{array}
$$

● ii) Boundary Conditions: All the ε_i chosen should satisfy the Essential boundary conditions of the problem. They may or may not satisfy the Natural boundary conditions.

● iii) Continuity: The functions chosen should be continuous at least upto the order required by the variational statement being used (usually C^1 continuous).

The ε_i fulfilling the above requirements constitute an "Admissible Set". The Ritz technique has been used in the present work.

▸ b) The Galerkin technique:

In this technique, too, the variables are expressed in the form given by eq(2.2). However, here, the Differential equation is used instead of the variational statement.

If L denotes an operator, we have

$$R = L\left(\sum_{i=1}^{M} c_i \, \varepsilon_i \right) \quad \ldots\ldots\ldots \quad (2.8)$$

where R is the error or residual due to the finite approximation. This error is now minimised by forcing it to be orthogonal to each of the basis ε_i, thereby reducing each of its components along ε_i to a minimum,

i.e $$\int_{\upsilon} R\,\varepsilon_i\,d\upsilon = 0 \qquad i = 1,2,\ldots,M \quad ..(2.9)$$

The completeness condition as mentioned previously in the Ritz analysis applies here also, in choosing an 'Admissible Set'. However, the functions have to satisfy all the boundary conditions, essential and natural. Continuity requirement is also generally higher and has to correspond at least to the order of the highest derivative of the Differential equation used.

Normally the choice of the basis set $\{\varepsilon_i\}$ is arbitrary. However, an Admissible Set of orthonormal functions can be very elegantly generated using the Gram-Schmidt procedure. This also ensures all the requirements of completeness, boundary conditions and continuity automatically.

[2.1.6] The Gram-Schmidt Procedure:

Given a set of linearly independent vectors:

$$l_1(s), l_2(s), \ldots, l_M(s)$$

one can construct a set of orthonormal vectors

$$g_1(s), g_2(s), \ldots, g_M(s)$$

by the Gram-Schmidt procedure as follows:

$$g_1 = \frac{l_1}{||l_1||}$$

$$p_2 = l_2 - \langle g_1 . l_2 \rangle\, g_1$$

$$g_2 = \frac{p_2}{||p_2||}$$

$$p_3 = l_3 - \langle g_1 . l_3 \rangle\, g_1 - \langle g_2 . l_3 \rangle\, g_2$$

$$g_3 = \frac{p_3}{||p_3||}$$

and so on, where

$$||l_1|| = \text{norm of } l_1 = \sqrt{\int_{s_1}^{s_2} l_1 . l_1 \, ds}$$

$$\langle g_1 . l_2 \rangle = \text{scalar or inner product of vectors } g_1 \text{ and } l_2$$

$$= \int_{s_1}^{s_2} g_1 . l_2 \, ds$$

[2.1.6.1] A Geometric interpretation:

For $M <= 3$, an interesting geometric interpretation can be easily given to the Gram-Schmidt procedure. All the vectors

reduce to the common three dimensional vectors of the R^3 Euclidean space.

Refering to fig (2.2a), by subtracting from l_2, its projection on g_1, we obtain the orthonormal vector g_2. Vector g_2 is now normal to g_1. Here,

$||l_1||$ gives the length of vector l_1, and

$\langle g_1 . l_2 \rangle$ is the cosine of the angle between g_1 and l_2.

A similar argument can be extended to explain fig (2.2b) for three dimensions.

[2.2] Laminated Composite Plate Analysis :

A plate is a solid continuum with a regular geometry and with one of its dimensions being very small compared to the other two lateral dimensions.

[2.2.1] Classical Plate Theory:

The Classical Plate Theory (henceforth called CPT) has been used in the present analysis. Plates are conventionally classified as thick plates ($a/h < 20$) or thin plates ($a/h >= 20$). CPT is the small deflection theory of bending of thin plates and it involves the following assumptions.

- The mid-surface displacements are small compared to the thickness of the plate.

- The mid-surface of the plate is unstrained and is the neutral plane after loading.
- Plane sections normal to the mid-surface remain plane and perpendicular to the midplane (implies negligible transverse shear strains e_{xz} and e_{yz}).
- σ_z can be neglected.

In accordance with CPT, the following displacement field has been assumed.

$$u(x,y,z) = u_o(x,y) - z\,\frac{\partial w_o}{\partial x}$$

$$v(x,y,z) = v_o(x,y) - z\,\frac{\partial w_o}{\partial y} \qquad \ldots \qquad (2.11)$$

$$w(x,y,z) = w_o(x,y)$$

[2.2.2] Strain Dispalcement Relations:

Using the above field, we have the following relations for the strains,

$$e_x = \frac{\partial u}{\partial x} = \frac{\partial u_o}{\partial x} - z\,\frac{\partial^2 w_o}{\partial x^2}$$

$$e_y = \frac{\partial v}{\partial y} = \frac{\partial v_o}{\partial y} - z\,\frac{\partial^2 w_o}{\partial y^2} \qquad \ldots\ldots\ldots(2.12)$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x} - 2z\,\frac{\partial^2 w_o}{\partial x\,\partial y}$$

[2.2.3] Stress-Strain Equations:

A laminate is made up a number of laminae. Each lamina has, in general, fibers oriented at a given angle. Thus the individual laminae are orthotropic in their own plane with the principal planes of orthotropy being oriented along and normal to the fiber direction.

If L and T stand for the fiber direction and its normal, we have the following constitutive relations for the k^{th} lamina,

$$\begin{Bmatrix} \sigma_L \\ \sigma_T \\ \tau_{LT} \end{Bmatrix}_K = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} e_L \\ e_T \\ \gamma_{LT} \end{Bmatrix} \qquad \ldots\ldots(2.13)$$

Since the L/T directions vary with each lamina, and that in general they do not coincide with the structural axes, the stresses and strains of each lamina will have to be transformed along the structural axes. Following the procedure given in Ref. [26] the transformed constitutive equations in terms of the reference axes is given by

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_K = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_K \begin{Bmatrix} e_x \\ e_y \\ \gamma_{xy} \end{Bmatrix}_K \qquad (2.14)$$

[2.2.4] Stress Resultants:

The laminate stress and moment resultants, as shown in Fig.(2.3), are defined as follows:

$$N_x,\ N_y,\ N_{xy} = \int_{-h/2}^{h/2} (\sigma_x,\ \sigma_y,\ \tau_{xy})\ dz$$

$$M_x,\ M_y,\ M_{xy} = \int_{-h/2}^{h/2} (\sigma_x,\ \sigma_y,\ \tau_{xy})\ Z\ dz \qquad \text{.....(2.15)}$$

Substituting relations (2.14) in (2.15), we have

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & & & & & \\ A_{12} & A_{22} & & \text{sym} & & \\ A_{16} & A_{26} & A_{66} & & & \\ B_{11} & B_{12} & B_{16} & D_{11} & & \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} e^o_x \\ e^o_y \\ \gamma^o_{xy} \\ \varkappa_x \\ \varkappa_y \\ \varkappa_{xy} \end{Bmatrix} \qquad \text{..........(2.15a)}$$

where, the $\varkappa$'s are curvatures and,

$$(A_{ij},\ B_{ij},\ D_{ij}) = \sum_{n=1}^{NL} \int_{Z_n}^{Z_{n+1}} \bar{Q}^{(n)}_{ij}(1,\ Z,\ Z^2)dz \qquad \text{.......(2.16)}$$

NL : Number of layers in laminate.

[2.2.5] Potential and Kinetic energies of the laminate:

In the absence of applied loads and body forces, the potential energy of the plate, in terms of the stresses and strains, can be written as follows:

$$V = \frac{1}{2} \int_0^a \int_0^b \int_{-h/2}^{+h/2} (\sigma_x e_x + \sigma_y e_y + \sigma_z e_z + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz})\, dx\, dy\, dz \quad \text{......(2.17)}$$

where

a = length of plate

b = width of plate

h = thickness of plate

Non-dimensionalising the variables x and y with respect to the plate dimensions, we have

$\xi = x/a$

$\eta = y/b$

Using eqns. (2.14) to (2.15a) and (2.17), we obtain the potential energy in the non-dimensional form as below:

$$V = \frac{abE_T}{2a^4} \int_0^1 \int_0^1 \left[a^2 A_{11} \left(\frac{\partial u_o}{\partial \xi} \right)^2 + a^2\mu^2 A_{22} \left(\frac{\partial v_o}{\partial \eta} \right)^2 + \right.$$

$$a^2 A_{66} \left(\frac{\partial v_o}{\partial \xi} \right)^2 + a^2\mu^2 A_{66} \left(\frac{\partial u_o}{\partial \eta} \right)^2 + 2a^2\mu A_{66} \left(\frac{\partial u_o}{\partial \eta} \frac{\partial v_o}{\partial \xi} \right) +$$

$$2a^2\mu A_{12}\left[\frac{\partial u_o}{\partial\xi}\frac{\partial v_o}{\partial\eta}\right] + 2a^2\mu A_{16}\left[\frac{\partial u_o}{\partial\xi}\frac{\partial u_o}{\partial\eta}\right] + 2a^2A_{16}\left[\frac{\partial u_o}{\partial\xi}\frac{\partial v_o}{\partial\xi}\right] +$$

$$2a^2\mu^2A_{26}\left[\frac{\partial u_o}{\partial\eta}\frac{\partial v_o}{\partial\eta}\right] + 2a^2\mu\ A_{26}\left[\frac{\partial v_o}{\partial\xi}\frac{\partial v_o}{\partial\eta}\right] - 2aB_{11}\left[\frac{\partial u_o}{\partial\xi}\frac{\partial^2 w_o}{\partial\xi^2}\right] -$$

$$2a\ \mu^3B_{22}\left[\frac{\partial v_o}{\partial\eta}\frac{\partial^2 w_o}{\partial\eta^2}\right] - 4a\ \mu^2B_{66}\left[\frac{\partial u_o}{\partial\eta}\frac{\partial^2 w_o}{\partial\xi\ \partial\eta}\right] -$$

$$4a\mu\ B_{66}\left[\frac{\partial v_o}{\partial\xi}\frac{\partial^2 w_o}{\partial\xi\,\partial\eta}\right] - 2a\ \mu^2B_{12}\left[\frac{\partial u_o}{\partial\xi}\frac{\partial^2 w_o}{\partial\eta^2}\right] -$$

$$2a\mu\ B_{12}\left[\frac{\partial v_o}{\partial\eta}\frac{\partial^2 w_o}{\partial\xi^2}\right] - 2a\mu\ B_{16}\left[2\frac{\partial u_o}{\partial\xi}\frac{\partial^2 w_o}{\partial\xi\ \partial\eta} + \frac{\partial u_o}{\partial\eta}\frac{\partial^2 w_o}{\partial\xi^2}\right] -$$

$$2a\ B_{16}\left[\frac{\partial v_o}{\partial\xi}\frac{\partial^2 w_o}{\partial\xi^2}\right] - 2a\mu^3B_{26}\left[\frac{\partial u_o}{\partial\eta}\frac{\partial^2 w_o}{\partial\eta^2}\right] - 2a\mu^2\ B_{26}$$

$$\left[2\frac{\partial v_o}{\partial\eta}\frac{\partial^2 w_o}{\partial\xi\ \partial\eta} + \frac{\partial v_o}{\partial\xi}\frac{\partial^2 w_o}{\partial\eta^2}\right] + D_{11}\left[\frac{\partial^2 w_o}{\partial\xi^2}\right]^2 + \mu^4D_{22}\left[\frac{\partial^2 w_o}{\partial\eta^2}\right] +$$

$$4\ \mu^2D_{66}\left[\frac{\partial^2 w_o}{\partial\xi\ \partial\eta}\right]^2 + 2\ \mu^2D_{12}\left[\frac{\partial^2 w_o}{\partial\xi^2}\ \ \frac{\partial^2 w_o}{\partial\eta^2}\right] +$$

$$4\ \mu\ D_{16}\left[\frac{\partial^2 w_o}{\partial\xi^2}\frac{\partial^2 w_o}{\partial\xi\ \partial\eta}\right] + 4\ \mu^3D_{26}\left[\frac{\partial^2 w_o}{\partial\eta^2}\ \ \frac{\partial^2 w_o}{\partial\xi\ \partial\eta}\right]\Bigg]\ d\xi\ d\eta$$

.........(2.18)

$d\xi\ d\eta$: differential area of plate mid-surface

$\mu = a/b$ = plate aspect ratio.

The kinetic energy of the plate in its non-dimensional form is obtained as:

$$T = \frac{\rho ab}{2} \int_{0}^{1} \int_{0}^{1} \int_{-h/2}^{+h/2} [(\dot{u})^2 + (\dot{v})^2 + (\dot{w})^2] \, d\xi \, d\eta \, dz \quad \ldots\ldots(2.19)$$

where, ρ is the density of the material.

The Ritz series expansion of the field variables in eq. (2.11) was carried out with the following orthonormal functions:

$$u_o(\xi,\eta) = C^u_{11}\, \alpha_{\xi_1}\, \beta_{\eta_1} + C^u_{12}\, \alpha_{\xi_1}\, \beta_{\eta_2} + \ldots.$$

$$v_o(\xi,\eta) = C^v_{11}\, \beta_{\xi_1}\, \alpha_{\eta_1} + C^v_{12}\, \beta_{\xi_1}\, \alpha_{\eta_2} + \ldots. \qquad (2.20)$$

$$w_o(\xi,\eta) = C^w_{11}\, \phi_{\xi_1}\, \phi_{\eta_1} + C^w_{12}\, \phi_{\xi_1}\, \phi_{\eta_2} + \ldots.$$

The following recurrence formulae as proposed by Bhat[17] which are based on the Gram-Schmidt procedure, were used to generate the above functions:

$$\bar{\alpha}_{\xi_2} = (\xi - b_2)\, \alpha_{\xi_1}$$

$$\ldots\ldots(2.21)$$

$$\bar{\alpha}_{\xi_f} = (\xi - b_f)\, \alpha_{\xi_{f-1}} - c_f \alpha_{\xi_{f-2}}$$

In general, a pair of constants (b_f, c_f) are required to build the f^{th} term and they are determined using the following orthogonality condition

$$\int_0^1 \alpha_{\xi_i} \alpha_{\xi_j} d\xi = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases}$$

which finally leads to the expressions for the constants as given below:

$$b_f = \int_o \xi \alpha^2_{\xi_{f-1}} d\xi, \quad C_f = \sqrt{\int_0^1 \alpha^2_{\xi_{f-1}} d\xi}$$

$$\alpha_{\xi_f} = \text{constant} \times \bar{\alpha}_{\xi_f}$$

where the constant is a normalising factor and is given by

$$\text{constant} = \frac{1}{\sqrt{\int_0^1 \alpha^2_{\xi_f} d\xi}}$$

The same set of equations (2.21) were used to generate the ξ, η functions for all the variables. These functions depend solely on the boundary conditions being used.

[2.2.6] Boundary Conditions:

The following boundary conditions were used in the present analysis:

- i) For Anti-symmetric cross-ply laminates

$$S2 \quad : \quad w = 0, \quad M_n = 0, \quad N_n = 0, \quad u_t = 0$$

$$C2 \quad : \quad w = 0, \quad w_{,n} = 0, \quad N_n = 0, \quad u_t = 0$$

- ii) For Ani-symmetric Angle-ply laminates

$$S3 \quad : \quad w = 0, \quad M_n = 0, \quad u_n = 0, \quad N_{nt} = 0$$

$$C3 \quad : \quad w = 0, \quad w_{,n} = 0, \quad u_n = 0, \quad N_{nt} = 0$$

where,

S => simply - supported

C => clamped

n => direction normal to edge being considered

t => direction tangential to edge

,n => derivative with respect to variable along direction n.

One among the many advangates of this method is the relative case and elegance with which starting terms for the Ritz series may be chosen. The starting terms of all variables satisfy both the geometric and the natural boundary conditions whereas the succeeding terms satisfy only the geometric

boundary conditions:

The following starting terms were chosen for the boundary conditions indicated:

S2, C2 :

$$\alpha_{\xi_1} = \text{constant } (\xi - \xi^2)$$

$$\beta_{\eta_1} = \text{constant } (0.006 - 2\eta^3 + 3\eta^2)$$

$$\beta_{\xi_1} = \text{constant } (0.006 - 2\xi^3 + 3\xi^2) \qquad (2.22)$$

$$\alpha_{\eta_1} = \text{constant } (\eta - \eta^2)$$

S3, C3 :

α and β were interchanged to give S3, C3 boundary conditions.

The ϕ functions were chosen on the lines of Bhat[17]. These functions satisfy the equivalent beam boundary conditions (both geometric and natural).

$$\text{S}: \quad \phi_{\xi_1} = \text{constant } (\xi - 2\xi^3 + \xi^4)$$

$$\text{C}: \quad \phi_{\xi_1} = \text{constant } (\xi^2 - 2\xi^3 + \xi^4) \qquad (2.23)$$

The ϕ_η functions are identical to the ϕ_ξ functions with η replacing ξ. All the constants above are the normalising factors of the respective terms as explained earlier.

[2.3] Hamilton's Principle:

Hamilton's Principle is a generalisation of the principle of virtual displacements to the dynamics of systems of particles, rigid bodies and deformable bodies. The Euler-Lagrange equations derived from this principle result in the familiar Newtonian equations of motion for a generalised set of co-ordinates.

The principle states that `of all possible paths a system could travel, from its position at time t_1 to another position at time t_2, the actual path is that path which renders the integral

$$I_o = \int_{t_1}^{t_2} L \, dt$$

an extremum. That is

$$\delta I_o = \delta \int_{t_1}^{t_2} L \, (q_1, q_2, \dot{q}_1, \dot{q}_2 \ldots) \, dt = 0 \qquad \ldots\ldots(2.24)$$

where L is the Lagrangian function. From eq (2.24) we also get the Euler-Lagrange equations as follows:

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0, \; i=1,2,3.. \qquad \ldots (2.25)$$

q_i are the generalised coordinates

dot represents derivative with respect to time.

For synchronous motion due to free vibration, we have:

$$u = u\, e^{j\omega t}$$
$$v = v\, e^{j\omega t} \qquad \ldots\ldots (2.26)$$
$$w = w\, e^{j\omega t}$$

where the system vibrates with a natural frequency ω.

Using equations (2.11), (2.18), (2.19), (2.20) and (2.26), we obtain the Lagrangian function L, as

$$L = T - V$$

Taking a variation on the set of unknown coefficients we have,

$$\delta \int_{t_1}^{t_2} L\, dt = 0 \qquad \ldots\ldots (2.27)$$

Factoring the time integral $\int_{t_1}^{t_2} e^{\,2j\omega t}\, dt$ out of the above equation and performing the necessary differentiations with respect to the unknown coefficients, we arrive at the familiar eigenvalue problem

$$[\,K\,]\,\{\,C\,\} = \lambda\, [\,M\,]\,\{\,C\,\} \qquad \ldots (2.28)$$

where λ is the eigenvalue and the non-dimensional frequency parameter is given by

$$\sqrt{\lambda} = \omega\, a^2\, (\rho/E_T h^2)^{1/2} \qquad \ldots (2.29)$$

K and M are the stiffness and mass matrices respectively and C

stands for the vector of unknown coefficients as also the eigen vector.

The number of terms (denoted by `noft'), used in the series of eq (2.20) for all the variables, in both the ξ and η directions are the same. Thus the final size of the stiffness and mass matrices can be shown to be $3 \times noft^2$.

[2.3.1] Components of the Stiffness and Mass Matrices:

The following type of notation used for the various sub-matrices that go to build the stiffness and mass matrices needs some explanation.

$$J = [\alpha^i \beta^j] \cdot [\phi^m \psi^n]$$

choosing a value of noft = 2 (i.e. two terms) to explain the above notations, we arrive at the following 4x4 matrices

$$J = \frac{1}{2}\int_0^1\int_0^1 \left[\begin{array}{cc|cc} \alpha_1^i & \beta_1^j & \alpha_1^i & \beta_2^j \\ \hline \alpha_2^i & \beta_1^j & \alpha_2^i & \beta_2^j \end{array}\right] \bullet \left[\begin{array}{cc|c} \phi_1^m \psi_1^n & \phi_1^m \psi_2^n & \text{-do-} \\ \phi_2^m \psi_1^n & \phi_2^m \psi_2^n & \\ \hline \multicolumn{2}{c|}{\text{— do —}} & \text{-do-} \end{array}\right] d\xi\, d\eta$$

All sub-matrices occur as products of a pair of matrices as shown above. The dot indicating the product DOES NOT stand for the usual matrix multiplication but implies that the corresponding individual elements of the two matrics should be multiplied. Further, all the elements of the first matrix are always, functions of the variable ξ only and those of the second matrix that of η only. The super scripts i, j, m and n

stand for the order of differentiation of that particular term. All the submatrices are of size $noft^2 \times noft^2$ and the sub-submatrices are of size noft x noft.

The stiffness and mass matrices can now be expressed in terms of the various sub-matrices using the notation explained above. So we have for the stiffness matrix,

$$[K] = \begin{bmatrix} [S_{11}] & [S_{12}] & [S_{13}] \\ & [S_{22}] & [S_{23}] \\ \text{SYM} & & [S_{33}] \end{bmatrix}$$

where

$$[S_{11}] = (A_{11}[a1] + A_{66}[a6_u] + A_{16}[a16_u])$$

$$[S_{12}] = (A_{12}[a12] + A_{66}[a6_{uv}] + A_{16}[a16_{uv}] + A_{26}[a26_{uv}])$$

$$[S_{22}] = (A_{22}[a2] + A_{66}[a6_v] + A_{26}[a26_v])$$

$$[S_{13}] = (B_{11}[b1] + B_{66}[b6_u] + B_{12}[b12_u] + B_{16}[b16_u] + B_{26}[b26_u])$$

$$[S_{23}] = (B_{22}[b2] + B_{66}[b6_v] + B_{12}[b12_v] + B_{16}[b16_v] + B_{26}[b26_v])$$

$$[S_{33}] = (D_{11}[d1] + D_{22}[d2] + D_{66}[d6] + D_{12}[d12] + D_{16}[d16] + D_{26}[d26])$$

Details of all of the above matrices are given in the Appendix.

The mass matrix may be expressed as follows:

$$[M] = \begin{bmatrix} [I] & & \text{SYM} \\ 0 & [I] & \\ 0 & 0 & [M_3] \end{bmatrix}$$

where $[I]$ stands for the identity matrix, and

$$[M_3] = [I] + \frac{1}{12}\Big[[\phi_\xi^1 \ \phi_\xi^1] \cdot [\phi_\eta^0 \ \phi_\eta^0] + [\phi_\xi^0 \ \phi_\xi^0] \cdot [\phi_\eta^1 \ \phi_\eta^1] \Big]$$

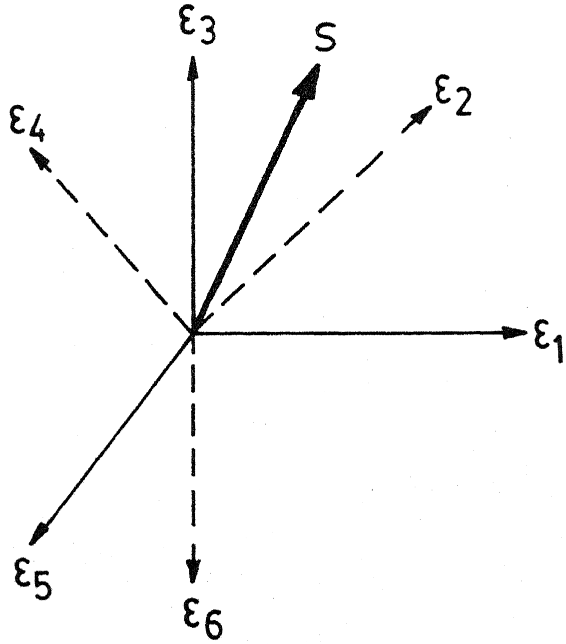
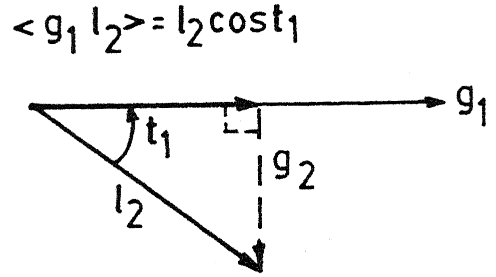
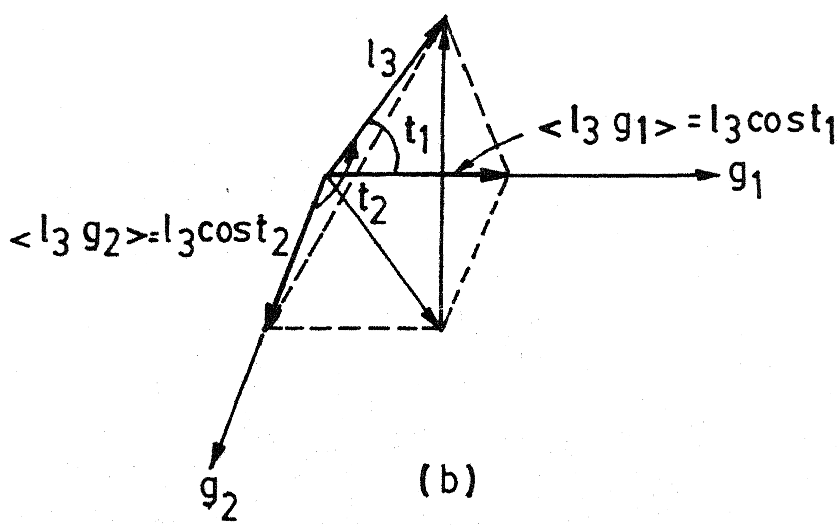


FIG. 2·1 CONCEPTUAL REPRESENTATION OF SOLUTION VECTOR 'S' IN THE CONFIGURATION SPACE



(a)



(b)

FIG. 2·2 GEOMETRIC INTERPRETATION OF THE GRAM-SCHMIDT PROCEDURE IN 2-D AND 3-D SPACE

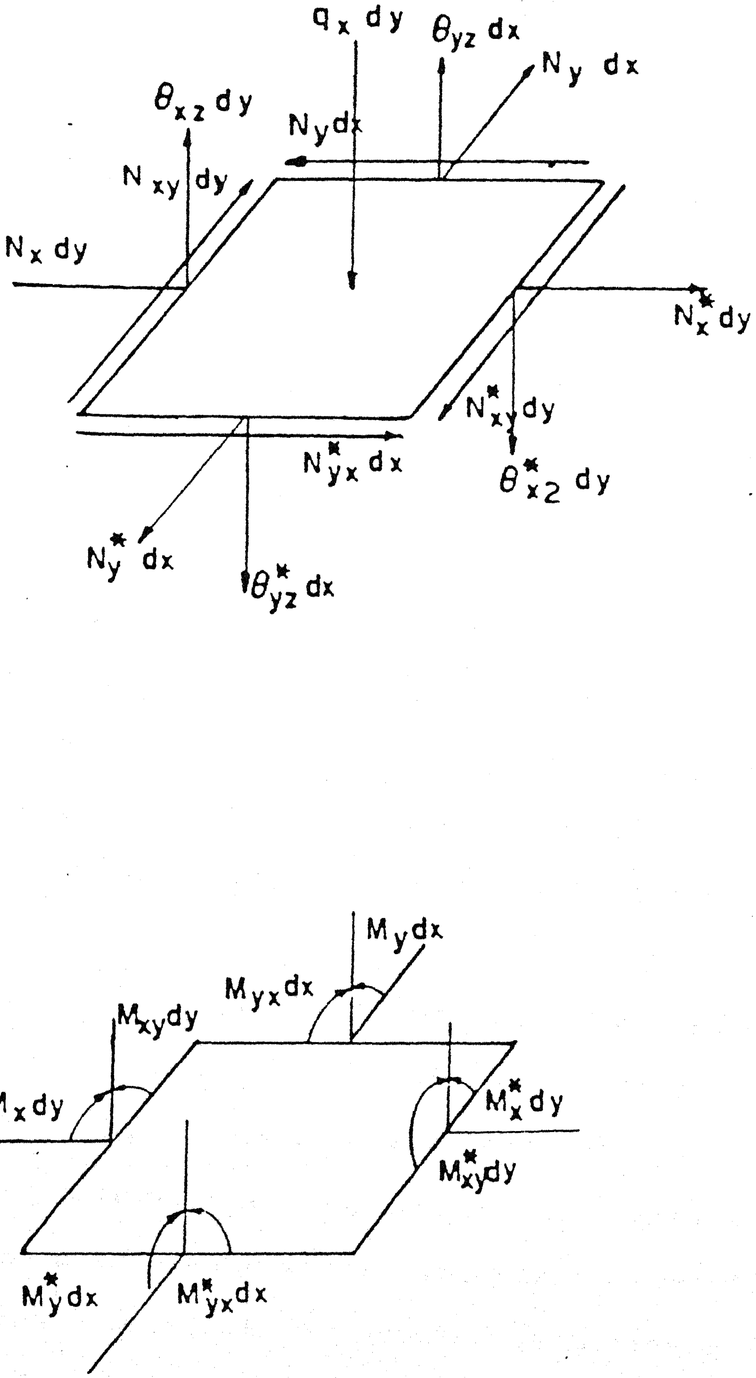


Fig. 2.3 STRESS RESULTANTS OF A LAMINATE

CHAPTER 3

RESULTS AND DISCUSSIONS

[3.1] Introduction:

The governing equations derived in Chapter 2 have been solved for free vibration analysis of composite plates. A computer program has been developed on the Mainframe HP 9000/850s using available NAG routines for the eigenvalue computations.

The core of any dynamic analysis (such as the present free vibration analysis) lies in the generation of the various stiffness and mass matrices. However, in the present analysis, automatic generation of the Ritz series expansion functions formed an equally important part, being inputs to the stiffness and mass elements.

A computer program incorporating the function generation aspect was developed to analyse out of plane vibrations of an isotropic plate and the results were checked with those available in the literature. The program was then modified to suit product integrals involving cross products of functions of different displacement variables. Sub-routines to generate the A,B and D matrices for the composite plate were also developed to produce the final program. This program was rechecked for numerical accuracies when reduced to the isotropic case and was then used for composite plates. The boundary conditions were

incorporated by changing only the starting terms of the series to obtain results for the new conditions. Finally the program was further modified to obtain results for an isotropic clamped tapered plate.

In order to perform the necessary integrations and differentiations for generating the various matrix elements, Gauss Quadrature and Numerical differentiation schemes were employed. The number of Gauss points used in the Gauss Quadrature integration scheme determine the highest order of the polynomial that can be integrated accurately. For example, if ng is the number of Gauss points used, a polynomial of order $\leq$ 2*n-1 can be integrated very accurately. In the present analysis, the order of the polynomial and the number of terms are related, i.e., with every new term, the increase in the order of the polynomial is by one. Consequently, the number of terms and the number of Gauss points also get related. Only 4 terms were used (number of terms = noft = 4) in the analysis as it was found that this was sufficient for convergence of the first three to four frequencies. Ten Gauss points were used for the integration. The total number of integrations for the most general laminate (i.e., assuming that the A,B and D matrices are all fully populated) would be $70 * noft^2$. The average total time for execution of one run of the program was less than a minute.

[3.2] Results:

The composite materials used in the present analysis are typical of High and Low modulus Graphite/Epoxy variety. Their properties have been presented in Table 3.1.

[3.2.1] Study of Convergence:

The size of the stiffness and mass matrices is given by $3*noft^2$. As mentioned previously, all results have been obtained using noft = 4. This leads to a final matrix size of 48 x 48 for the stiffness and mass matrices.

Convergence studies of the non-dimensional frequency $\sqrt{\lambda}$ were carried out for isotropic, orthotropic and composite plates. Figure 3.1 shows a typical behaviour for the case of a simply-supported, square, anti-symmetric angle-ply laminate. It is evident from the figure that 4 terms are sufficient to ensure convergence upto the 4^{th} mode. It was also observed that the rate of convergence exhibited slight oscillations.

[3.2.2] Effect of Side/thickness ratio:

Figure 3.2 shows the variation of the non-dimensional frequency with various side/thickness ratios for a simply-supported, square, anti-symmetric angle-ply laminate. The near flat response seen, is to be expected, since the Classical Plate Theory neglects transverse shear effects. It

generally over predicts the response but is fairly accurate for thin plates ($a/h >= 20$). Thus for thick plates, -where shear effects are predominant- the results are conservative.

[3.2.3] Isotropic and Orthotropic plates:

Table 3.2 shows a comparison of the non-dimensional frequency for a simply-supported, square isotropic plate. Solving for these out of plane frequencies required a value of noft = 4, for convergence of the first 4 frequencies but the matrix sizes were only 16 x 16. As can be seen from the table, the results by the present method are in excellent agreement with those of Bhat[17] and Liew[21]. Frequencies for various aspect ratios were also checked. It may be pointed out that the the choice of the starting functions and the generation of the succeeding terms is much simpler in the present method as compared to that due to Liew[21]. The constant 0.006 in the starting terms of the Ritz series shown in Chapter 2 needs explanation. The constant was introduced to prevent the complementary in-plane displacement from going to zero at the specified edge. The numerical value of 0.006 for the constant was arrived at after a number of trials with various other values were carried out, for minimising the frequencies.

In Table 3.3 are shown frequencies of an orthotropic, square, simply-supported plate obtained by employing the same functions that were used for the isotropic plate studies. It is observed that the results obtained by this technique are in

close agreement with those seen in Ref [25].

[3.2.4] Symmetric Cross-ply laminates:

Figures 3.3 and 3.4 show the variation of the non-dimensional frequency, $\sqrt{\lambda}$ with aspect ratio for four layered symmetric cross-ply laminates with simply-supported and clamped boundary conditions along the four edges, respectively. The frequencies for the clamped case are seen to be higher than the corresponding frequencies for the simply-supported case. This is due to the fact that the clamped boundary conditions tend to increase the overall stiffness of the system thereby requiring higher energies to set it vibrating. The frequencies are also seen to be increasing with aspect ratio. This indicates that the geometry of the plate also has an influence on the frequency. Physically it maybe explained as follows. The fundamental or lowest frequencies of a plate can be attributed mainly to the out of plane vibration modes. This is because the in-plane stiffnesses are higher compared to the flexural stiffness and hence the in-plane frequencies also higher. Now, the sub-matrix for out of plane stiffness is a direct function of the aspect ratio. Hence it is evident that for low aspect ratios', this dominant sub-matrix (dominant as far as the lower frequencies go) has a fairly low value and the corresponding fundamental frequencies are also low. Naturally the frequencies would increase for higher aspect ratios. It is also seen that the frequencies for the low modulus material are

lower compared to that of the high modulus material. This is simply because the plate of the higher modulus material is stiffer.

[3.2.5] Anti-symmetric Cross-ply laminates:

The variation of non-dimensional frequency with aspect ratio for anti-symmetric cross-ply laminates with simply-supported and clamped boundary conditions are shown in figures 3.5 and 3.6. The frequencies for this case are seen to be lower compared to the ones for the symmetric cross-ply laminates. This behaviour maybe explained by considering the fact that for symmetric laminates, the elements of the B matrix are all zeros. Since the B matrix couples in-plane and bending displacements, it tends to make the plate more flexible. Thus, when the B matrix is zero, the stiffness is increased and consequently frequencies also increase for such laminates. The variation with aspect ratio and material is seen to be similar to the previous case as the behaviour is the same as explained earlier.

[3.2.6] Anti-symmetric angle-ply laminates:

Figures 3.7 to 3.12 show the variation of the non-dimensional frequency with aspect ratio for anti-symmetric

angle-ply laminates with fiber orientation of 15°,30° and 45° and for simply-supported and clamped boundary conditions. The nature of variation of frequencies with aspect ratio and boundary conditions are seen to be the same as in the case of cross-ply laminates and can be explained in a similar manner. However it is interesting to study the frequencies of plates with aspect ratios < 1 having different fiber orientations. It is seen that for low aspect ratios the frequencies decrease with increase with fiber orientation angle i.e., $\omega_{15}>\omega_{30}>\omega_{45}$. This is because for a given value of aspect ratio, there is a preferential increase in stiffness along the shorter length of the plate for small fiber orientation angles, which makes it more difficult to excite such plates. However for fiber orientation angle of 45°, no such preferred direction exists and the structure is more flexible. But for aspect ratios ≥ 1 the trend reverses.

Figures 3.13 to 3.16 show the variation of non-dimensional frequency with fiber orientation angle for simply-supported and clamped plates of different materials. It is known that as the number of layer are increased the bending-extension coupling (due to the b_{16} and b_{26} elements of the B matrix) decreases. This increases the frequency of vibration. In figures 3.13 and 3.14, it is clearly seen that for plates with more than two layers, the frequency increases with fiber orientation angle as expected. However, for two layered plates there is a decrease due to the dominating effect of coupling. In figures 3.15 and 3.16, the decoupling effect is evident with the increase in

number of layers, but the trends for the differently layered plates are all the same due to the clamped boundary condition.

Table 3.4 shows a comparison of non-dimensional frequencies for anti-symmetric angle-ply laminates with 45° fiber orientation. This table reflects the superior behaviour of the present method. It is seen that the results obtained are in excellent comparison with those of Ref [12]. It would be appropriate to note that the results presented in Ref [12] were through an FEM analysis that included shear deformation. In both the cases the results have been obtained for a/h = 50, which is close to thin plates. The results indicate that the present method is quite sufficient for the study of slightly thick plates.

[3.2.7] Tapered plates:

The isotropic analysis was modified to include a tapered plate with the taper on the thickness being along X direction only. A clamped boundary condition was used. Table 3.5 shows a comparison of frequencies for a square plate with different taper parameters. The taper parameter chosen is the slope of the taper. It is seen that the results by the present method match with those of Ref[24] very well (within 4% error). The present method lends itself very well to a Ritz type of analysis with simpler starting terms as compared to the Galerkin, used in Ref[24], wherein the starting terms are constrained to satisfy all the boundary conditions.

TABLE 3.1

PROPERTIES OF COMPOSITE MATERIALS USED

Properties	Material-A	Material-B
$\dfrac{E_L}{E_T}$	40	25
$\dfrac{G_{LT}}{E_T}$	0.5	0.5
ν_{LT}	0.25	0.25

TABLE 3.2

COMPARISON OF NON-DIMENSIONAL FREQUENCIES

$$\sqrt{\lambda} = \omega a^2 \sqrt{\frac{\rho h}{D}}$$

FOR A SIMPLY-SUPPORTED SQUARE ISOTROPIC PLATE

Mode	Exact Value[17]	Bhat[17]	Liew et al[21]	Present Method
1	19.739	19.739	19.74	19.738
2	49.348	49.348	49.35	49.347
3	49.348	49.348	-	49.347
4	78.957	78.957	79.03	78.956
5	98.696	99.304	99.25	99.303
6	98.696	99.304	-	99.303

$\nu = 0.3$

TABLE 3.3

COMPARISON OF NATURAL FREQUENCIES FOR A SIMPLY-SUPPORTED ORTHOTROPIC+ SQUARE PLATE

Mode	Ref(25)	Present Method
1	0.0497	0.04966
2	0.1120	0.11205
3	0.1354	0.13544
4	0.1987	0.19874

+ $E_L/E_T = 1.9$, $G_{LT}/E_T = 0.56$, $\nu_{LT} = 0.44$

TABLE 3.4

COMPARISON OF NATURAL FREQUENCIES OF A SIMPLY-SUPPORTED LAMINATE OF MATERIAL "A" ($45^\circ/-45^\circ/45^\circ/-45^\circ$), FOR VARIOUS ASPECT RATIOS

Aspect Ratio	F.E.M[12]	Present Method
0.2	9.816	9.336
0.6	15.689	15.458
1.0	24.343	24.109
1.2	29.321	29.102
1.6	40.653	40.414
2.0	53.989	53.688

$a/h = 50.$

TABLE 3.5

COMPARISION OF NATURAL FREQUENCIES OF CLAMPED SQUARE TAPERED ISOTROPIC PLATES FOR VARIOUS TAPER PARAMETERS tr.

Taper Parameter, tr	Ref. 24	Present Method
0.2	9.15	9.442
0.6	10.185	10.307
0.8	10.927	10.728

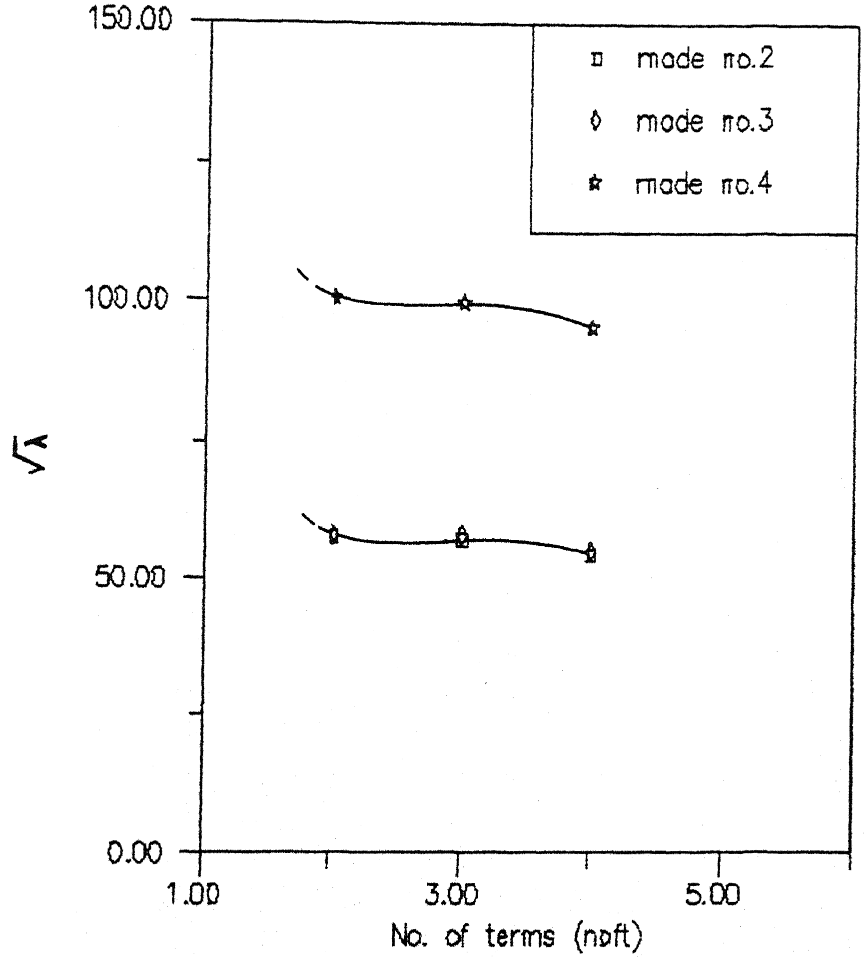


Fig.3.1 : Convergence pattern for a simply supported anti-symmetric angle ply laminate; 45/−45/45/−45; a/h = 50; aspect ratio = 1.

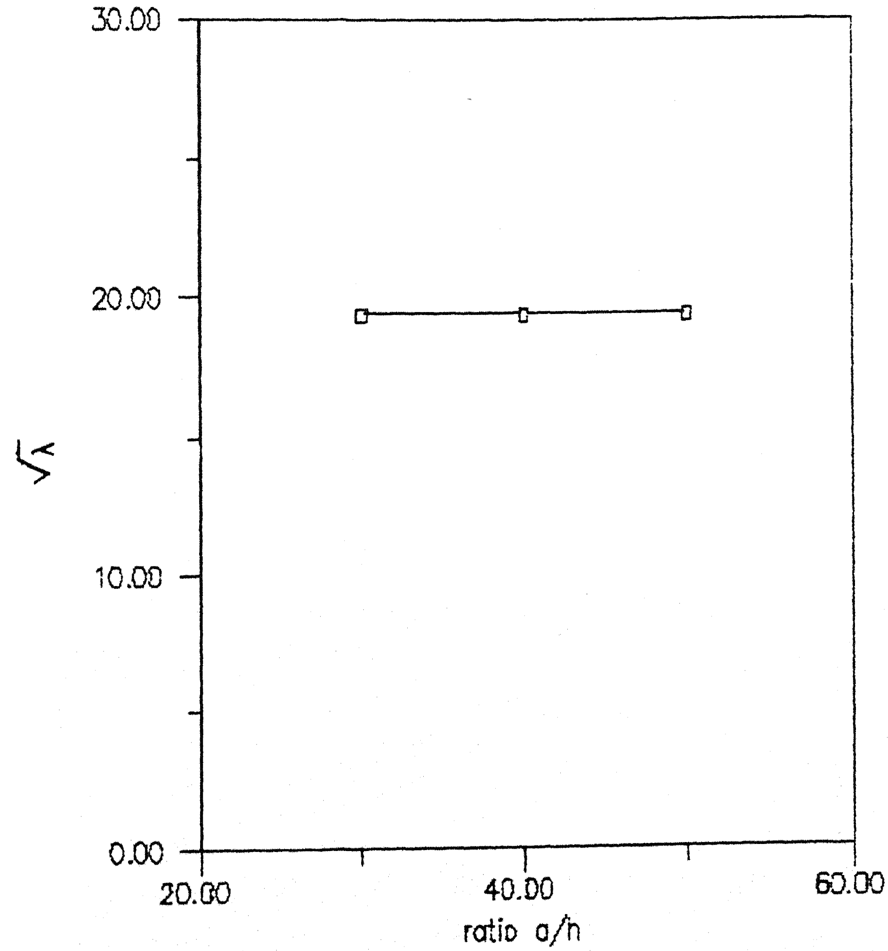


Fig.3.2 : Variation of non-dimensional frequency with side/thickness ratio for simply supported anti-symmetric angle-ply laminates of material B; 45/-45/45/-45

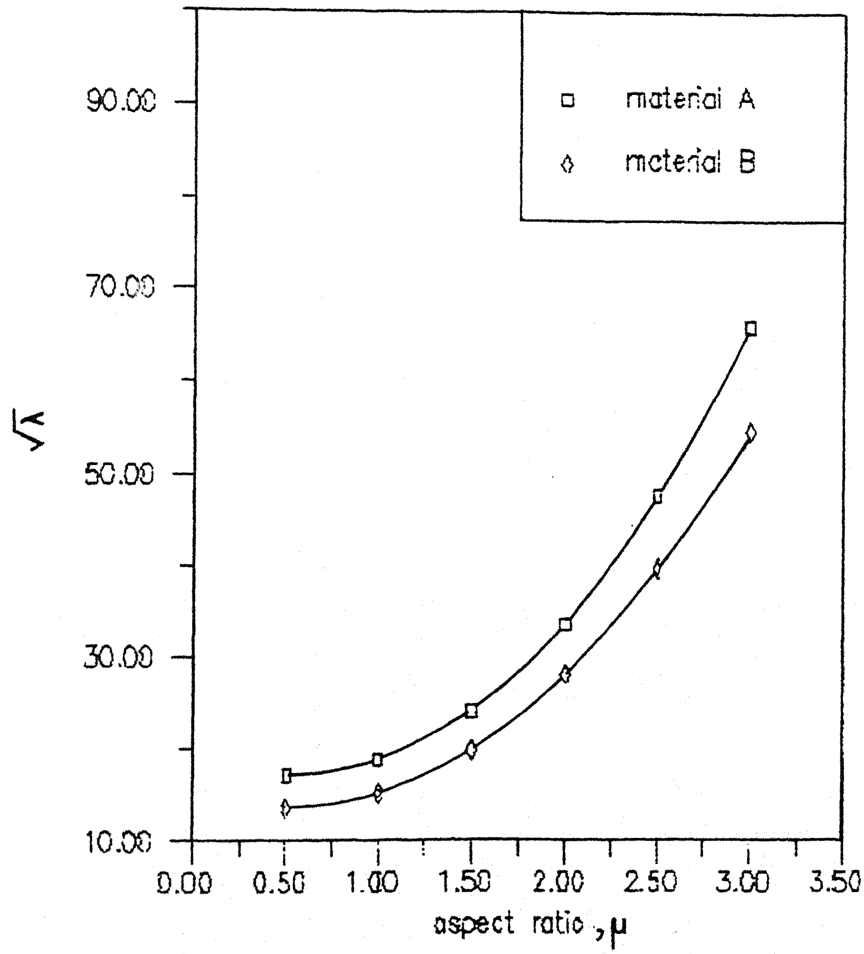


Fig.3.3 : Variation of non-dimensional frequency with aspect ratio for symmetric cross-ply laminates;0/90/90/0;(S.S.S.S);a/h=100.

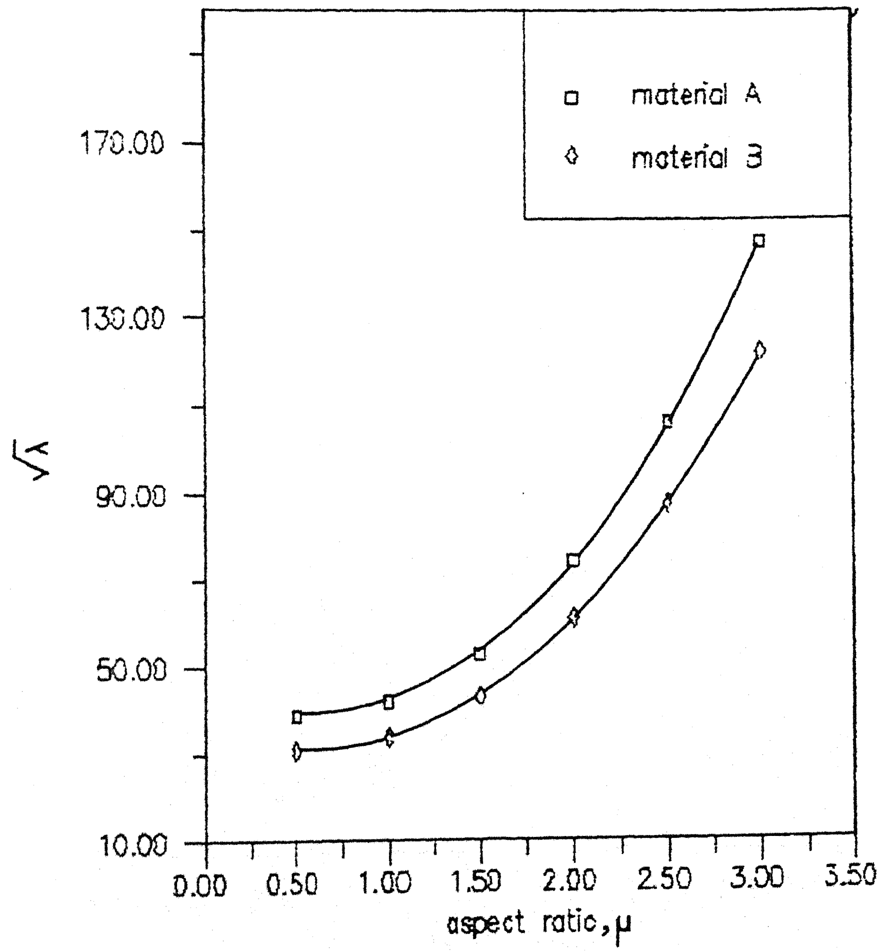


Fig.3.4 : Variation of non-dimensional frequency with aspect ratio for symmetric cross-ply laminates;0/90/90/0;(C.C.C.C);a/h=100.

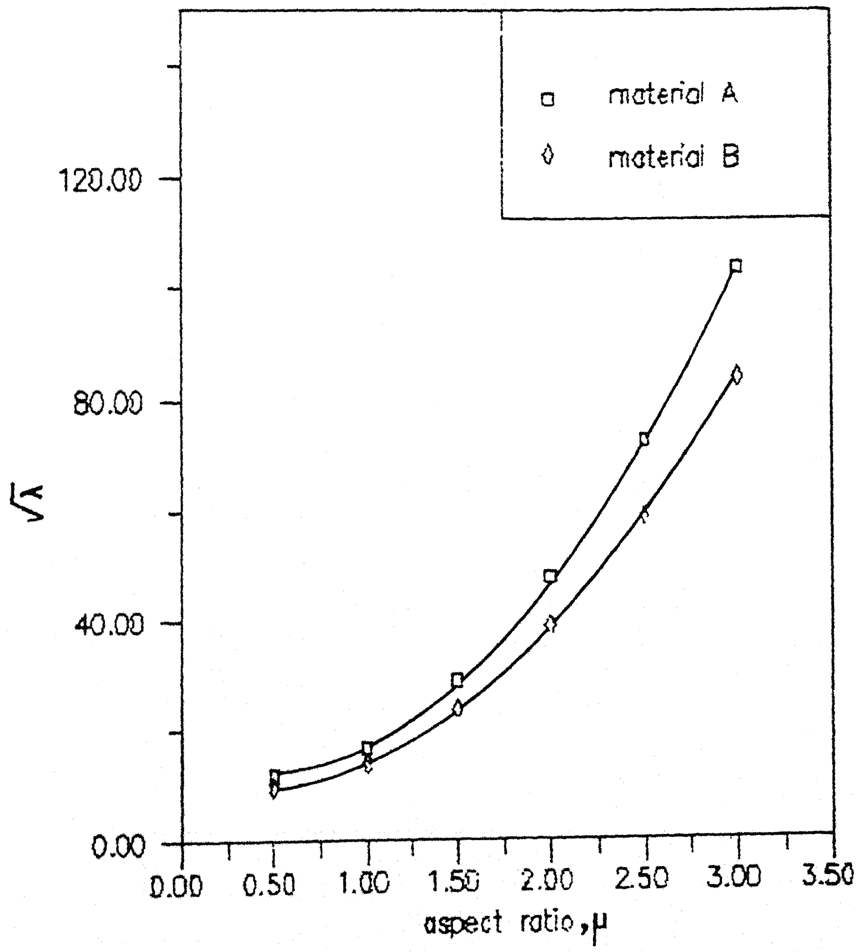


Fig.3.5 : Variation of non-dimensional frequency with aspect ratio for anti-symmetric cross-ply laminates;0/90/0/90:(S.S.S.S.);a/h=100

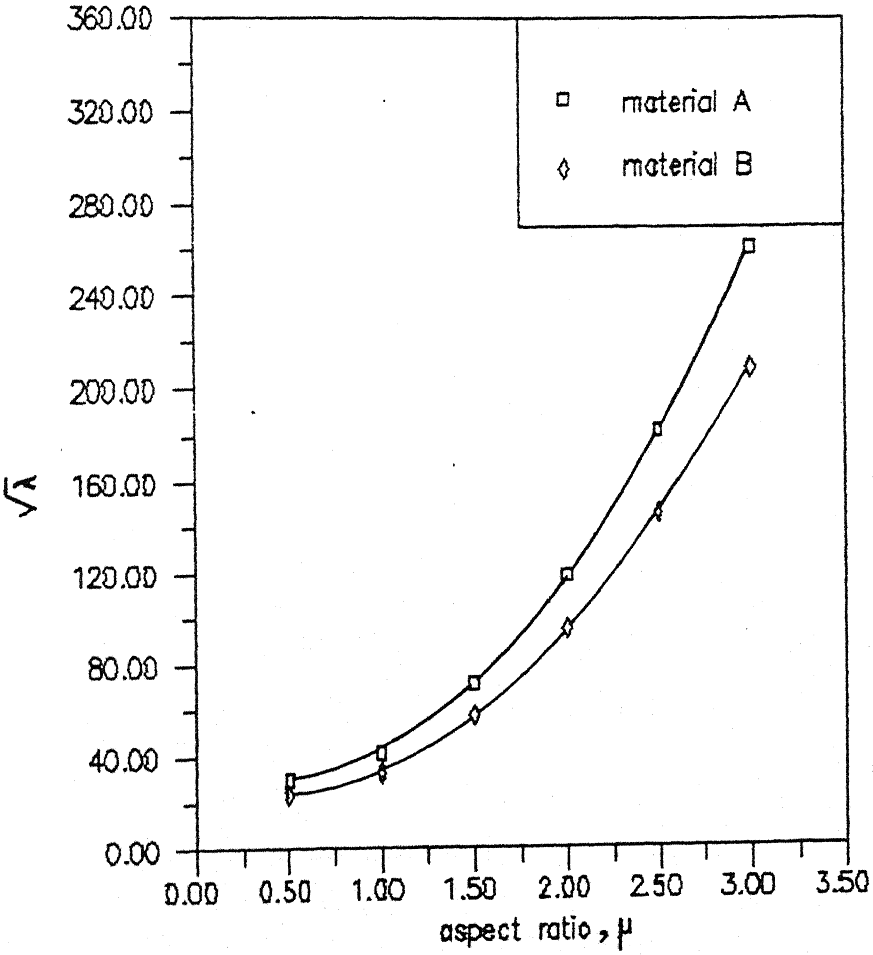


Fig.3.6 : Variation of non-dimensional frequency with aspect ratio for anti-symmetric cross-ply laminates;0/90/0/90;(C.C.C.C);a/h=100.

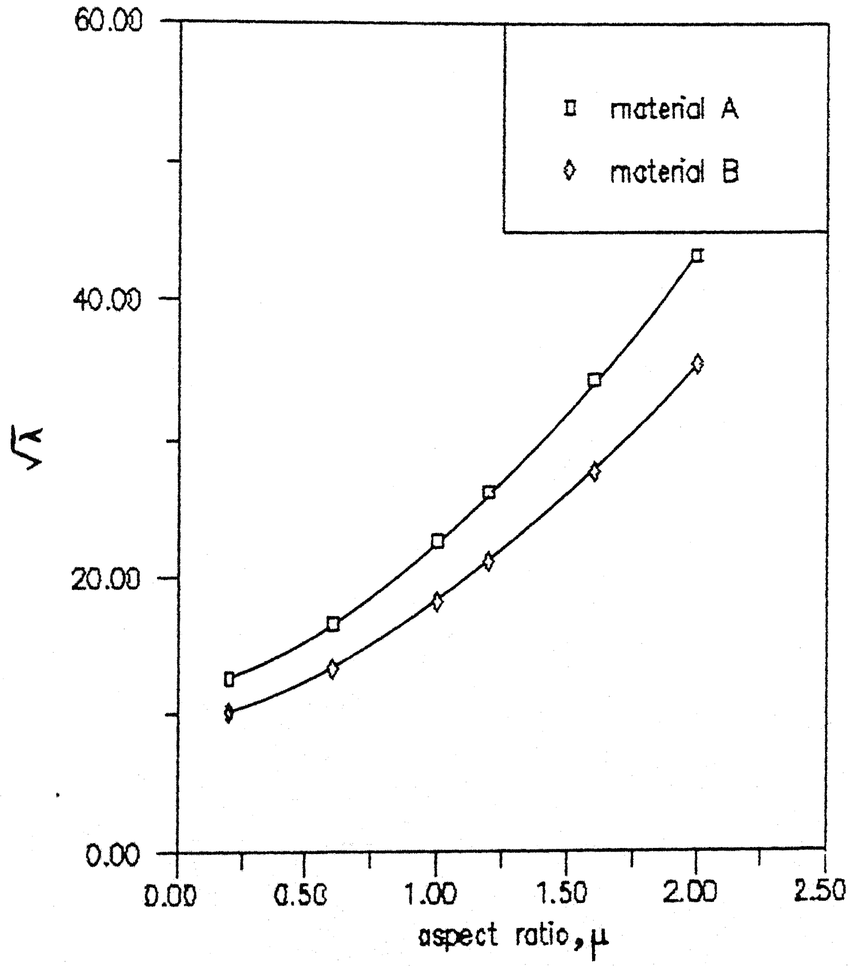


Fig.3.8 : Variation of non-dimensional frequency with aspect ratio for anti-symmetric angle-ply laminates;30/-30/30/-30;(S.S.S.S);a/h=50.

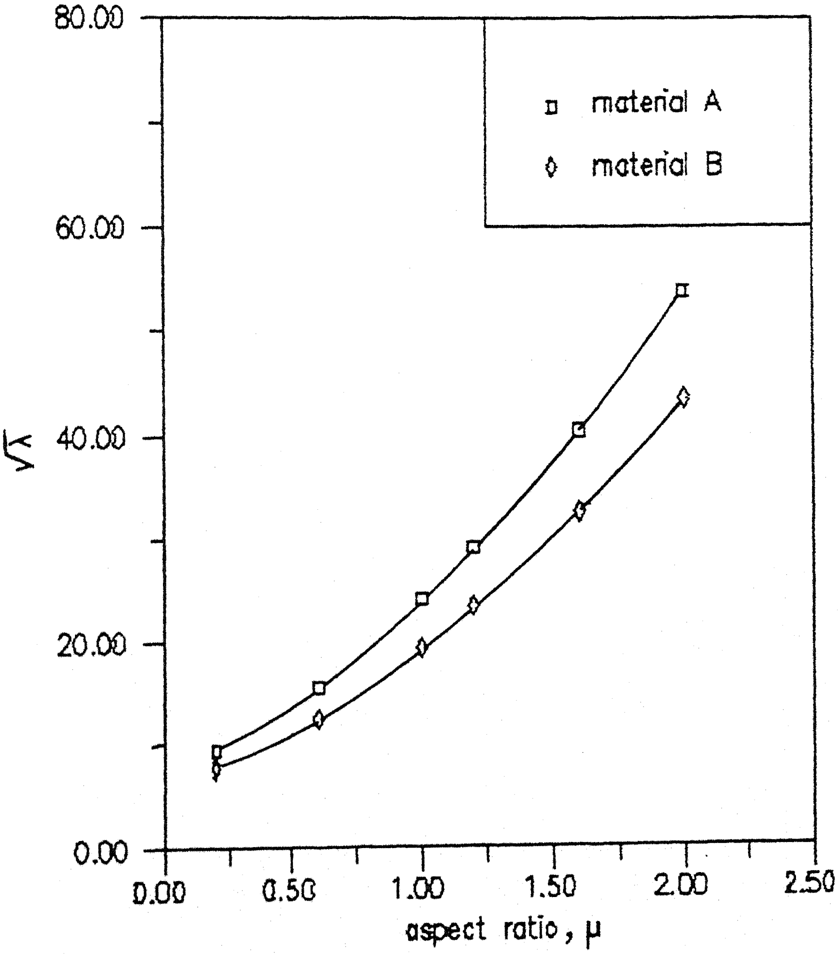


Fig.3.9 : Variation of non-dimensional frequency with aspect ratio for anti-symmetric angle-ply laminates;45/-45/45/-45;(S.S.S.S);a/h=50.

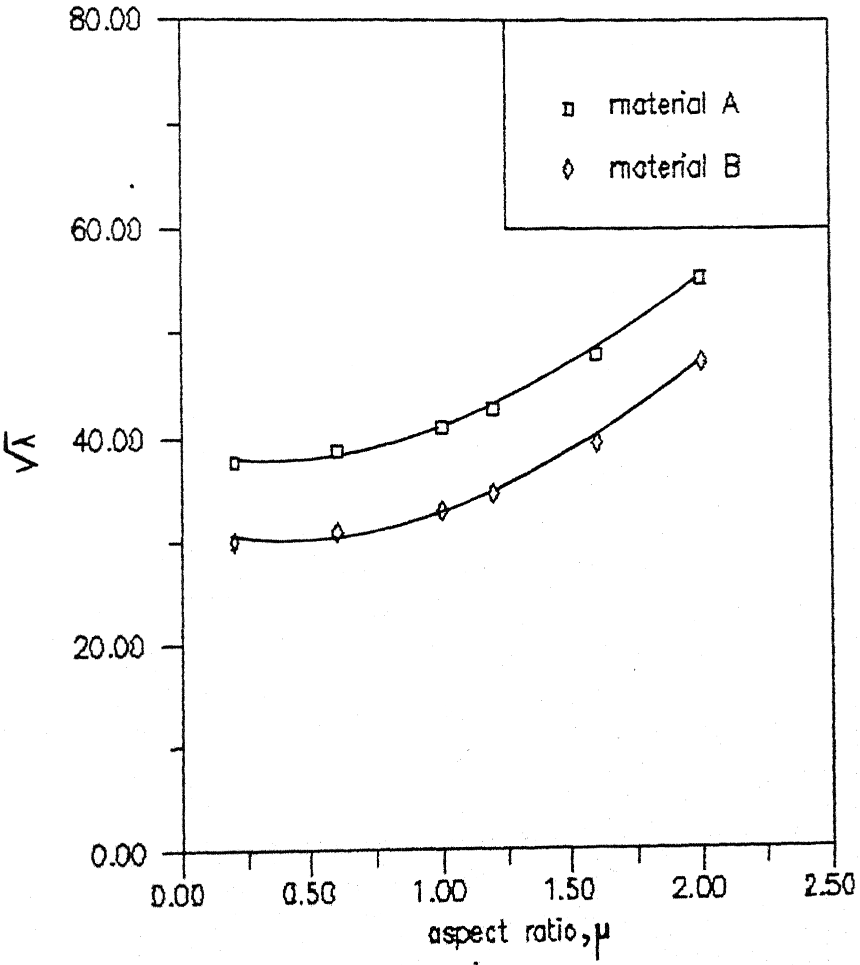


Fig.3.10 : Variation of non-dimensiuonal frequency with aspect ratio for anti-symmetric angle-ply laminates;15/-15/15/-15;(C.C.C.C);a/h=50.

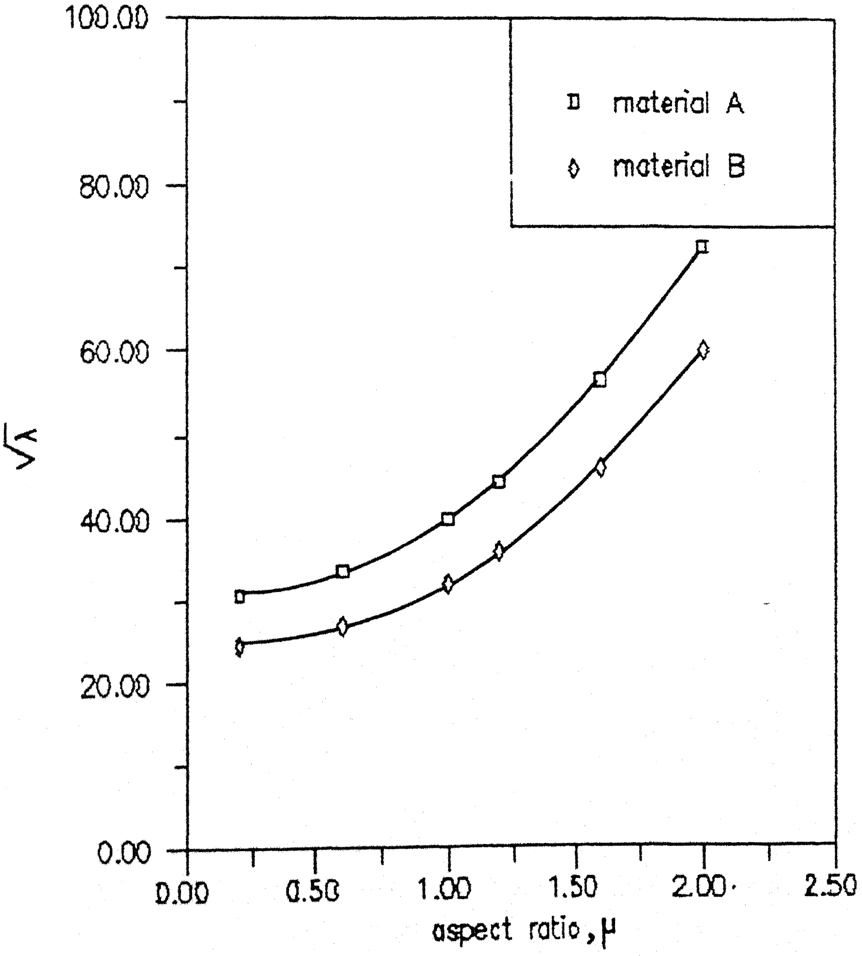


Fig.3.11 : Variation of non-dimensional frequency with aspect ratio for anti-symmetric angle-ply laminates;30/-30/30/-30;(C.C.C.C);a/h = 50.

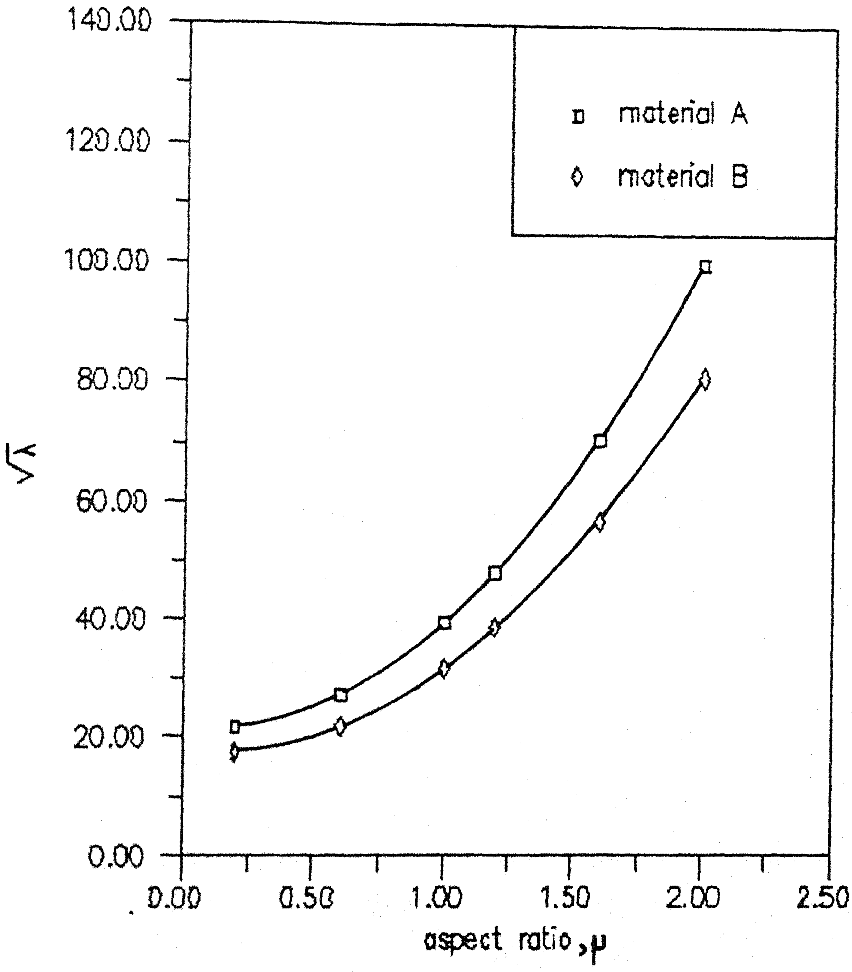


Fig.3.12 : Variation of non-dimensional frequency with aspect ratio for anti-symmetric angle-ply laminates;45/-45/45/-45;(C.C.C.C);a/h = 50.

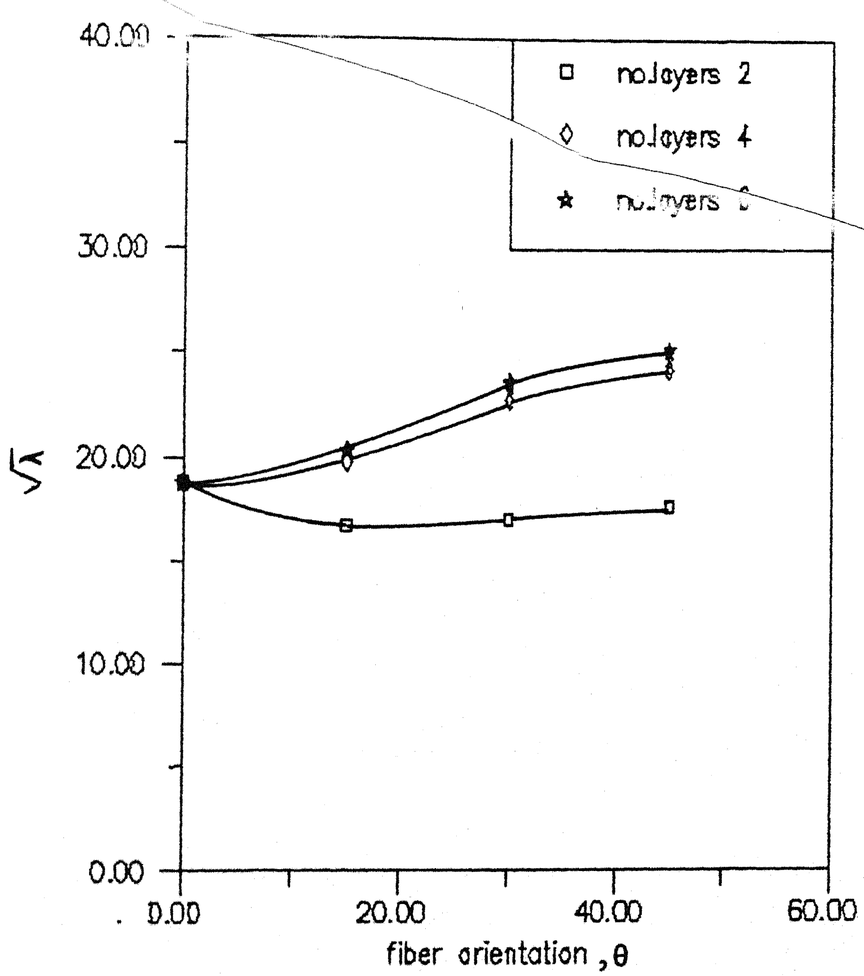


Fig.3.13 : Variation of non-dimensional frequency with theta for simply-supported square anti-symmetric angle-ply laminates of material A.

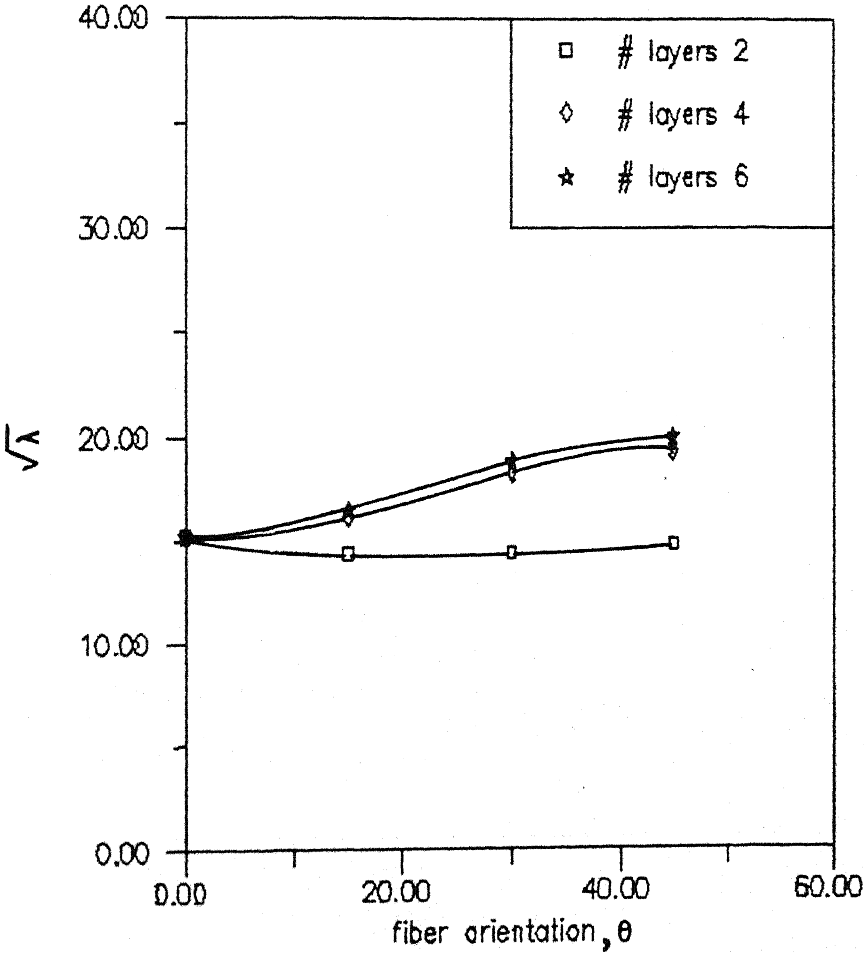


Fig.3.14 : Variation of non-dimensional frequency with theta for simply-supported square anti-symmetric angle-ply laminates of material B.

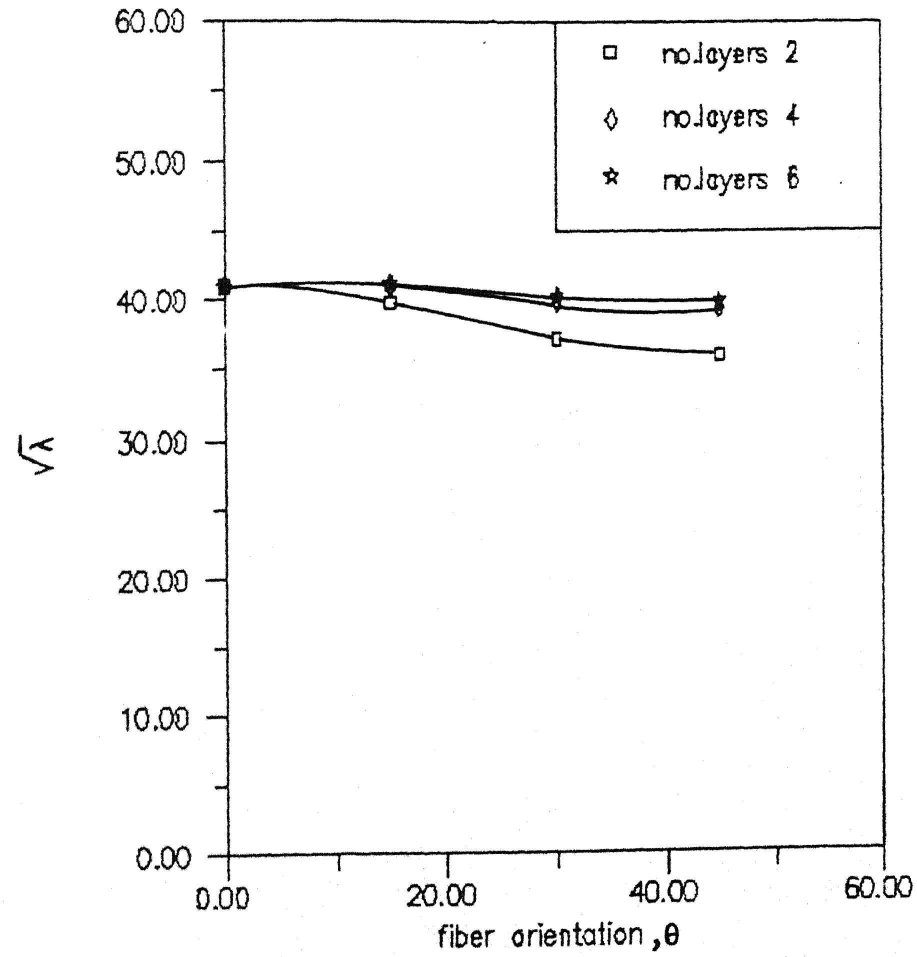


Fig.3.15 : Variation of non-dimensional frequency with theta for clamped square anti-symmetric angle-ply laminates of material A.

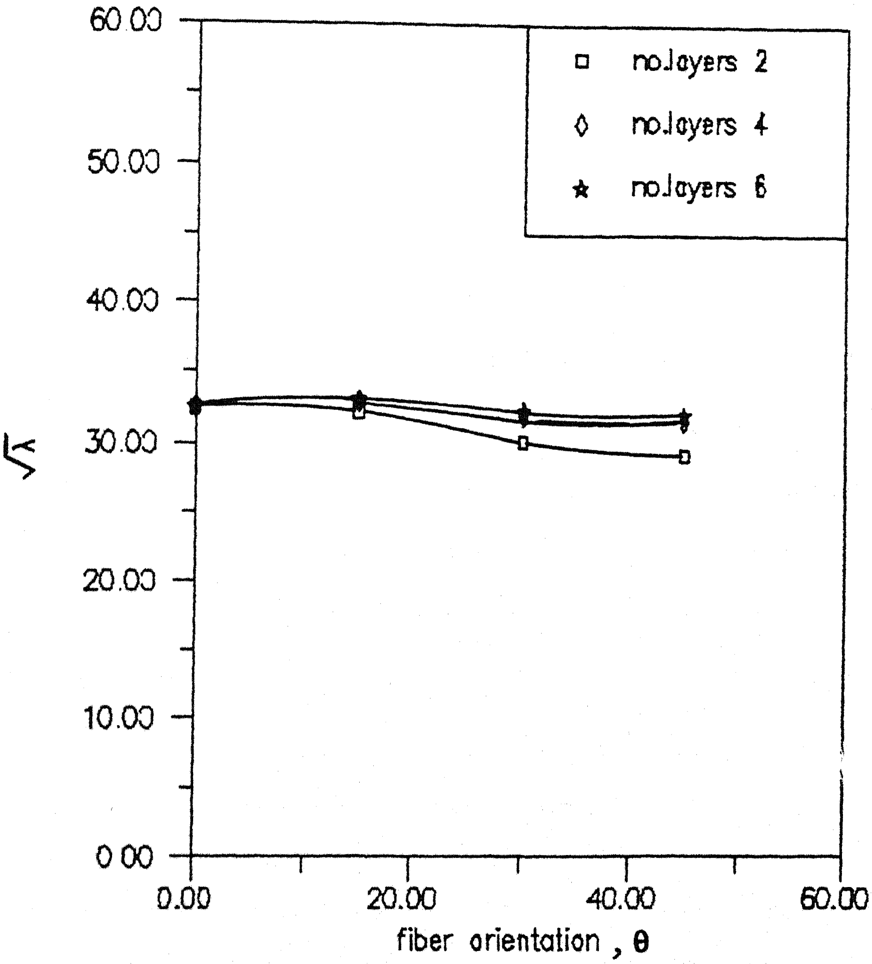


Fig.3.16 : Variation of non-dimensional frequency with theta for clamped square anti-symmetric angle-ply laminates of material B.

CHAPTER 4

CONCLUSIONS AND SUGGESTIONS FOR FUTURE WORK

4.1] Conclusions:

The following can be conclusively said about the present technique:

- The present formulation is amenable to easy and extremely rapid programming.
- The choice of the starting terms for the series is very easily made and the automatic generation and simple Gaussian integration possible, make the technique an elegant one.
- Convergence is rapid with a fewer number of terms, due to the orthogonalisation and this technique is excellent for a Ritz type analysis.

[4.2] Suggestions for future work:

- For a particular type of problem, the program can be optimised by studying the nature of boundary conditions and avoiding repetitions of the term generation and integration operations.
- The number of Gauss points to be used can be regulated by the number of terms being used in the series.
- The present technique can be further exploited for free vibration analysis of thick plates if shear deformation is included in the formulation.

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APPENDIX

Component matrices of the sub-matrices that make up the

iffness matrix:

$[a1]$: $a^2 [\alpha_\xi^1 \alpha_\xi^1] \cdot [\beta_\eta^0 \beta_\eta^0]$

$[a6_u]$: $a^2\mu^2 [\alpha_\xi^0 \alpha_\xi^0] \cdot [\beta_\eta^1 \beta_\eta^1]$

$[a16_u]$: $a^2\mu\Big[[\alpha_\xi^1 \alpha_\xi^0] \cdot [\beta_\eta^0 \beta_\eta^1] + [\alpha_\xi^0 \alpha_\xi^1] \cdot [\beta_\eta^1 \beta_\eta^0]\Big]$

$[a12]$: $a^2\mu [\alpha_\xi^1 \beta_\xi^0] \cdot [\beta_\eta^0 \alpha_\eta^1]$

$[a6_{uv}]$: $a^2\mu [\alpha_\xi^0 \beta_\xi^1] \cdot [\beta_\eta^1 \alpha_\eta^0]$

$[a16_{uv}]$: $a^2 [\alpha_\xi^1 \beta_\xi^1] \cdot [\beta_\eta^0 \alpha_\eta^0]$

$[a26_{uv}]$: $a^2\mu^2 [\alpha^0{}_\xi \beta^0{}_\xi] \cdot [\beta_\eta^1 \alpha_\eta^1]$

$[a2]$: $a^2\mu^2 [\beta_\xi^0 \beta_\xi^0] \cdot [\alpha_\eta^1 \alpha_\eta^1]$

$[a6_v]$: $a^2 [\beta_\xi^1 \beta_\xi^1] \cdot [\alpha_\eta^0 \alpha_\eta^0]$

$[a26_v]$: $a^2\mu\Big[[\beta_\xi^1 \beta_\xi^0] \cdot [\alpha_\eta^0 \alpha_\eta^1] + [\beta_\xi^0 \beta_\xi^1] \cdot [\alpha_\eta^1 \alpha_\eta^0]\Big]$

$[b1]$: $-a [\alpha_\xi^1 \phi_\xi^2] \cdot [\beta_\eta^0 \phi_\eta^0]$

$[b6_u]$: $-2a\mu^2 [\alpha_\xi^0 \phi_\xi^1] \cdot [\beta_\eta^1 \phi_\eta^1]$

$[b12_u]$: $-a\mu^2 [\alpha_\xi^1 \phi_\xi^0] \cdot [\beta_\eta^0 \phi_\eta^2]$

$[b16_u]$: $-a\mu\left[2[\alpha_\xi^1\ \phi_\xi^1]\ .\ [\beta_\eta^0\ \phi_\eta^1] + [\alpha_\xi^0\ \phi_\xi^2]\ .\ [\beta_\eta^1\ \phi_\eta^0]\right]$

$[b26_u]$: $-a\mu^3\ [\alpha_\xi^0\ \phi_\xi^0]\ .\ [\beta_\eta^1\ \phi_\eta^2]$

$[b2]$: $-a\mu^3\ [\beta_\xi^0\ \phi_\xi^0]\ .\ [\alpha_\eta^1\ \phi_\eta^2]$

$[b6_v]$: $-\ 2a\mu\ [\beta_\xi^1\ \phi_\xi^1]\ .\ [\alpha_\eta^0\ \phi_\eta^1]$

$[b12_v]$: $-a\mu\ [\beta_\xi^0\ \phi_\xi^2]\ .\ [\alpha_\eta^1\ \phi_\eta^0]$

$[b16_v]$: $-a\ \ [\beta_\xi^1\ \phi_\xi^2]\ .\ [\alpha_\eta^0\ \phi_\eta^0]$

$[b26_v]$: $-a\mu^2\left[2[\beta_\xi^0\ \phi_\xi^1]\ .\ [\alpha_\eta^1\ \phi_\eta^1] + [\beta_\xi^1\ \phi_\xi^0]\ .\ [\alpha_\eta^0\ \phi_\eta^2]\right]$

$[d1]$: $[\phi_\xi^2\ \phi_\xi^2]\ .\ [\phi_\eta^0\ \phi_\eta^0]$

$[d2]$: $\mu^2\ [\phi_\xi^0\ \phi_\xi^0]\ .\ [\phi_\eta^2\ \phi_\eta^2]$

$[d6]$: $4\mu^2\ [\phi_\xi^1\ \phi_\xi^1]\ .\ [\phi_\eta^1\ \phi_\eta^1]$

$[d12]$: $\mu^2\left[\ [\phi_\xi^2\ \phi_\xi^0]\ .\ [\phi_\eta^0\ \phi_\eta^2] + [\phi_\xi^0\ \phi_\xi^2]\ .\ [\phi_\eta^2\ \phi_\eta^0]\right]$

$[d16]$: $2\mu\left[\ [\phi_\xi^2\ \phi_\xi^1]\ .\ [\phi_\eta^0\ \phi_\eta^1] + [\phi_\xi^1\ \phi_\xi^2]\ .\ [\phi_\eta^1\ \phi_\eta^0]\right]$

$[d26]$: $2\mu^3\left[\ [\phi_\xi^0\ \phi_\xi^1]\ .\ [\phi_\eta^2\ \phi_\eta^1] + [\phi_\xi^1\ \phi_\xi^0]\ .\ [\phi_\eta^1\ \phi_\eta^2]\right]$